

# Schrödinger-Poisson equations with singular potentials in $\mathbb{R}^3$

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**ABSTRACT:** The existence and  $L^\infty$  estimate of positive solutions are discussed for the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + (\lambda + \frac{1}{|y|^\alpha})u + \phi(x)u = |u|^{p-1}u, & x = (y, z) \in \mathbb{R}^2 \times \mathbb{R}, \\ -\Delta \phi = u^2, & \lim_{|x| \rightarrow +\infty} \phi(x) = 0, y = (x_1, x_2) \in \mathbb{R}^2 \text{ with } |y| = \sqrt{x_1^2 + x_2^2}, \end{cases} \quad (0.1)$$

where  $\lambda \geq 0$ ,  $\alpha \in [0, 8)$  and  $\max\{2, \frac{2+\alpha}{2}\} < p < 5$ .

**Key words:** Schrödinger-Poisson equation; singular potential; nonnegative PS sequence; positive solution.

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## 1 Introduction

In this paper, we study the following type of Schrödinger-Poisson equations

$$\begin{cases} -\Delta u + V(x)u + \phi(x)u = |u|^{p-1}u, \\ -\Delta \phi = u^2, & \lim_{|x| \rightarrow +\infty} \phi(x) = 0, \quad x = (x_1, x_2, z) \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

where  $p \in (2, 5)$ , and the potential function  $V(x)$  is of the form

$$(V) \quad V_\lambda(x) = \lambda + \frac{1}{|y|^\alpha}, \quad \lambda \geq 0, \alpha \in [0, 8) \quad \text{and } |y| = \sqrt{x_1^2 + x_2^2}.$$

Problem (1.1) can be viewed as the stationary problem of the following coupled Schrödinger-Poisson system:

$$\begin{cases} i\psi_t - \Delta \psi + \phi(x)\psi = f(|\psi|)\psi, \\ -\Delta \phi = |\psi|^2, & \lim_{|x| \rightarrow +\infty} \phi(x) = 0, \quad x \in \mathbb{R}^3, \end{cases} \quad (1.2)$$

where  $f(|\psi|)\psi = |\psi|^{p-1}\psi + \omega_0\psi$ ,  $\omega_0 > 0$ ,  $2 < p < 5$  and  $\psi : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}$ . In fact, motivated by [8], we may seek a solution of (1.2) with the following type:

$$\psi(x, t) = u(x)e^{i(\eta(x) + \omega t)}, \quad u(x) \geq 0, \quad \eta(x) \in \mathbb{R}/2\pi\mathbb{Z}, \quad \omega \geq \omega_0.$$

Then, by (1.2),  $u$  should satisfy a system

$$\begin{cases} -\Delta u + (\omega - \omega_0 + |\nabla \eta(x)|^2)u + \phi(x)u = |u|^{p-1}u, \\ u\Delta \eta(x) + 2\nabla u \nabla \eta = 0, \\ -\Delta \phi = u^2, & \lim_{|x| \rightarrow +\infty} \phi(x) = 0, \quad x \in \mathbb{R}^3. \end{cases}$$

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Furthermore, similar to [6, 9], for  $x \in \mathbb{R}^3$ , if we let  $u(x) = u(y, z) = u(|y|, z)$  and

$$\eta(x) = \begin{cases} \arctan(x_2/x_1), & \text{if } x_1 > 0, \\ \arctan(x_2/x_1) + \pi, & \text{if } x_1 < 0, \\ \pi/2, & \text{if } x_1 = 0 \text{ and } x_2 > 0, \\ -\pi/2, & \text{if } x_1 = 0 \text{ and } x_2 < 0, \end{cases}$$

it is easy to see that  $\eta(x) \in C^2(\mathbb{R}^3 \setminus T_-)$ , where  $T_- := \{(x_1, x_2, z) \in \mathbb{R}^3 : x_1 = 0, x_2 \leq 0\}$ . By a simple calculation we know that

$$\Delta\eta(x) = 0, \quad \nabla\eta(x) \cdot \nabla u(x) = 0, \quad |\nabla\eta(x)| = \frac{1}{|y|^2}, \quad \text{for } x \in \mathbb{R}^3 \setminus T_-.$$

These show that  $u(|y|, z)$  is actually a nonnegative solution of (1.1) with  $\alpha = 2$  and  $\lambda = \omega - \omega_0$ . Furthermore,  $\psi(x)$  solves (1.2) with angular momentum:

$$M(\psi) = \operatorname{Re} \int_{\mathbb{R}^3} i\bar{\psi}x \wedge \nabla\psi dx = - \int_{\mathbb{R}^3} u^2x \wedge \nabla v(x) dx = -(0, 0, |u|_{L^2}^2).$$

For problem (1.1), more and more results have been published under various conditions on the potential function  $V(x)$  and on the nonlinear term  $|u|^{p-1}u$ , for examples, if  $V(x) = \text{const}$ , that is  $\alpha = 0$  in (V), the non-existence of nontrivial solution of (1.1) for  $p \notin (1, 5)$  was proved in [13] by a Pohozaev type identity, a radially symmetric positive solution was obtained in [11] and [14] for  $p \in [3, 5)$ , etc. It is known that we may find a nontrivial weak solution of problem (1.1) by looking for a nonzero critical point of the related variational functional of problem (1.1). It is also known that the weak limit of a so-called Palais-Smale sequence ((PS) sequence, in short) of the variational functional is usually a weak solution, but it may be a trivial solution unless we can prove that the variational functional satisfies the Palais-Smale condition ((PS) condition, in short), that is, a (PS) sequence has a strongly convergent subsequence. However, without condition (1.3) below, it seems very difficult to show a (PS) sequence converges strongly. In this paper, instead of trying to prove the (PS) condition, we adapt a trick used in [7], which is essentially a version of the concentration-compactness principle due to [22], to show directly that the weak limit of a (PS) sequence is indeed a nontrivial solution. For this purpose, we have to ensure that the (PS) sequence obtained by the deformation Lemma [24] is nonnegative and  $\phi(x)$  is bounded in  $D^{1,2}(\mathbb{R}^3)$ , this is because there is a term  $\phi(x)u$  appearing in problem (1.1), which is usually called a nonlocal term. As a by-product, in this paper we provide a simple approach for getting a nonnegative (PS) sequence and a bound of  $\phi(x)$  in  $D^{1,2}(\mathbb{R}^3)$ , see Lemma 2.6, this may be useful in certain situations. Note that in [5, 10, 12, 7] the authors studied the single stationary Schrödinger equation, that is, the first equation of (1.1) with  $\phi(x) = 0$  (i.e. without nonlocal term), in this case it is not necessary to seek a nonnegative (PS) sequence, see e.g. [5, 7]. It seems no any results for Schrödinger-Poisson system (1.1) under condition (V) with  $\alpha > 0$ . We should mention that our results of this paper cover the case of  $\alpha = 0$ , that is, the constant potential case. In this paper, we give also a priori estimate for solutions of (1.1), see Lemma 4.4, and get also a classical solution (except  $|y| = 0$ ) for (1.1) with  $\lambda = 0$ ,  $\alpha \in (0, 8)$  and  $\max\{2, \frac{2+\alpha}{2}\} < p < 5$ .

For problem (1.1) with constant potential, i.e. taking  $\alpha = 0$  in (V), the existence and nonexistence results were established by Ruiz in [21], he proved that (1.1) has always a positive radial solution if  $p \in (2, 5)$  and does not admit any nontrivial solution if  $p \leq 2$ . A ground state for (1.1) with  $p \in (2, 5)$  was proved in [4]. The existence of non-radially symmetric solution was shown in [15] and multiple solutions for (1.1) were obtained in [2, 11]. If the potential  $V(x)$  is not a constant, problem (1.1) has been studied in [4] for  $p \in (3, 5)$  and [26] for  $p \in (2, 3]$ . (1.1) with more general nonlinearities has been studied in [1, 3, 20, 23, 25], etc. To ensure that the variational functional of problem (1.1) satisfies the (PS) condition, in the papers [4],[26] the following conditions are assumed

$$V(x) \leq V_\infty = \liminf_{|x| \rightarrow \infty} V(x), \quad (1.3)$$

$$2V(x) + (\nabla V(x), x) \geq 0 \text{ a.e. } x \in \mathbb{R}^3. \quad (1.4)$$

It is clear that the above conditions are not true for the potential given by (V). So, we cannot follow the same tricks as that of [4, 26] to deal with problem (1.1). Without condition (1.4), it seems difficult even in

showing that a (PS) sequence is bounded in the working Sobolev space, specially in the case of  $p \in (2, 3)$ . Motivated by [6], here we try to find a bounded and nonnegative (PS) sequence directly from the well-known deformation Lemma ([24], Lemma 2.3).

Before stating our main results, we introduce some notations, definitions and recall some properties of the solution of the second equation (Poisson equation) in (1.1). For  $\alpha \geq 0$  and  $x = (y, z) \in \mathbb{R}^2 \times \mathbb{R}$ , define

$$E = \{u(x) \in D^{1,2}(\mathbb{R}^3) : u(x) = u(|y|, z) \text{ and } \int_{\mathbb{R}^3} \frac{u^2}{|y|^\alpha} dx < \infty\}, \quad (1.5)$$

and for  $\lambda > 0$ , we denote

$$H = \{u \in E : \lambda \int_{\mathbb{R}^3} u^2 dx < \infty\}.$$

Clearly  $H \subset E$ ,  $H \subset H^1(\mathbb{R}^3)$  and  $H$  is a Hilbert space, its scalar product and norm are given by

$$\langle u, v \rangle_H = \int_{\mathbb{R}^3} [\nabla u \nabla v + V_\lambda(x) uv] dx \quad \text{and} \quad \|u\|_H^2 = \langle u, u \rangle_H, \quad (1.6)$$

respectively, where  $V_\lambda(x) = \lambda + \frac{1}{|y|^\alpha}$ .

Throughout this paper, we denote the standard norms of  $H^1(\mathbb{R}^3)$  and  $L^p(\mathbb{R}^3)$  ( $1 \leq p \leq +\infty$ ) by  $\|\cdot\|$  and  $\|\cdot\|_p$ , respectively. Then, (1.6) implies that  $\|\cdot\|_H$  is an equivalent norm of  $\|\cdot\|$  if  $\alpha = 0$ .

By Lemma 2.1 of [21], we know that  $-\Delta\phi(x) = u^2$  has a unique solution in  $D^{1,2}(\mathbb{R}^3)$  with the form of

$$\phi(x) := \phi_u(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy, \quad \text{for any } u \in L^{\frac{12}{5}}(\mathbb{R}^3), \quad (1.7)$$

and

$$|\nabla\phi_u(x)|_2 \leq C|u|_{12/5}^2, \quad \int_{\mathbb{R}^3} \phi_u(x) u^2 dy \leq C|u|_{12/5}^4. \quad (1.8)$$

For  $\lambda > 0$  and  $u \in H$ , we can define the variational functional of problem (1.1) as follows:

$$I(u) := I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V_\lambda(x) u^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} dx. \quad (1.9)$$

Since (1.8),  $I_\lambda$  is well defined on  $H$  and  $I_\lambda \in C^1(H, \mathbb{R})$  with

$$(I'_\lambda(u), v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + V_\lambda(x) uv) dx + \int_{\mathbb{R}^3} \phi_u(x) uv dx - \int_{\mathbb{R}^3} |u|^{p-1} uv dx \quad (1.10)$$

for all  $v \in H$  with  $\lambda > 0$  and  $p \in (1, 5)$ . Furthermore, it is known that a weak solution of (1.1) corresponds to a nonzero critical point of the functional  $I$  in  $H$  if  $\lambda > 0$ .

However, if  $\lambda = 0$ , then  $H = E$ . In this case, (1.7) (1.8) are not always true for  $u \in E$ . Therefore, the integrations  $\int_{\mathbb{R}^3} |u|^p dx$ ,  $\int_{\mathbb{R}^3} \phi_u(x) u^2 dx$  and  $\int_{\mathbb{R}^3} \phi_u(x) uv dx$  may not be well defined for  $u, v \in E$ .

In this paper, we want to establish some existence results for problem (1.1) for both  $\lambda > 0$  and  $\lambda = 0$ . To this end, we set

$$T = \{x \in \mathbb{R}^3 : |y| = 0\} \text{ where } |y| = \sqrt{x_1^2 + x_2^2}. \quad (1.11)$$

Hence, by an approximation procedure, see Section 4, we can find a weak solution  $u \in E$  of (1.1) with  $\lambda = 0$  in the sense of

$$\int_{\mathbb{R}^3} (\nabla u \nabla \varphi + \frac{1}{|y|^\alpha} u \varphi) dx + \int_{\mathbb{R}^3} \phi_u(x) u \varphi dx = \int_{\mathbb{R}^3} |u|^{p-1} u \varphi dx, \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^3 \setminus T). \quad (1.12)$$

Note that  $\int_{\mathbb{R}^3} \frac{1}{|y|^\alpha} u \varphi dx$  may be not integrable for  $u \in E$  and  $\varphi \in C_0^\infty(\mathbb{R}^3)$ , this is why we take  $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$  above instead of  $\varphi \in C_0^\infty(\mathbb{R}^3)$ . So, it is reasonable for us to define a weak solution for (1.1) as follows.

**Definition 1.1.**  $u \in E \setminus \{0\}$  is said to be a weak solution of (1.1) with  $\lambda \geq 0$  if  $\phi_u \in D^{1,2}(\mathbb{R}^3)$  and  $u$  satisfies

$$\int_{\mathbb{R}^3} [\nabla u \nabla \varphi + (\frac{1}{|y|^\alpha} + \lambda)u\varphi]dx + \int_{\mathbb{R}^3} \phi_u(x)u\varphi dx = \int_{\mathbb{R}^3} |u|^{p-1}u\varphi dx \quad (1.13)$$

for all  $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$ .

We mention that the above definition also enables us to get a classical solution. In fact, if  $u \in E$  and  $\phi_u \in D^{1,2}(\mathbb{R}^3)$  satisfies (1.13), by using our Lemmas 4.2 and 4.3, as well as Theorems 8.10 and 9.19 in [16], we can prove that  $u \in C^2(\mathbb{R}^3 \setminus T)$ , that is,  $u$  is a classical solution of (1.1), see Theorem 3.1 in section 3.

For the following single Schrödinger equation

$$-\Delta u + \frac{u}{|y|^\alpha} = f(u), \quad x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N, \quad N \geq 3 \quad (1.14)$$

with  $|y| = \sqrt{\sum_{k=1}^{N+1-i} x_k^2}$ ,  $i < N$ , the authors of paper [5] proved that (1.14) has a nontrivial solution in  $H^1(\mathbb{R}^N)$  if  $\alpha = 2$ ,  $N > i \geq 2$  and  $f(t)$  is supposed to have some kinds of double powers behavior which ensure that  $F(u) = \int_0^u f(s)ds$  is well defined in  $L^1(\mathbb{R}^N)$  when  $u \in D^{1,2}(\mathbb{R}^N)$ . In [5], the authors used a variational method to seek first a nontrivial solution of (1.14) in  $D^{1,2}(\mathbb{R}^N)$ , then they proved this solution is in  $L^2(\mathbb{R}^N)$ . Formally, (1.14) is nothing but the first equation of problem (1.1) by taking  $\lambda = 0$ ,  $N = 3$  and getting rid of the nonlocal term  $\phi(x)u$ . However, even for  $f(u) = |u|^{p-1}u$  with  $p \in (2, 5)$ ,  $F(u)$  is not well defined in  $D^{1,2}(\mathbb{R}^N)$ , then the method and results of [5] do not work for our problem. For these reasons, it seems difficult to choose a working space to solve (1.1) directly if  $\lambda = 0$ . In this paper, we prove first that (1.1) has always a solution  $u_\lambda$  in  $H^1(\mathbb{R}^3)$  for each  $\lambda > 0$ , then show that  $\{u_\lambda\}$  (as a sequence of  $\lambda$ ) is bounded in  $E$ , as mention above we can finally use an approximation process to get a weak solution of (1.1) for  $\lambda = 0$  in the sense of (1.12).

The main results of this paper can be stated now as follows:

**Theorem 1.1.** Let  $\alpha \in [0, 8)$ ,  $\max\{2, \frac{2+\alpha}{2}\} < p < 5$  and condition (V) be satisfied. Then, problem (1.1) has at least a positive solution  $u_\lambda \in H \cap C^2(\mathbb{R}^3 \setminus T)$  for every  $\lambda > 0$ . Furthermore, if  $\lambda \in (0, 1]$ , there exists  $C > 0$  which is independent of  $\lambda \in (0, 1]$  such that the solution  $u_\lambda$  satisfies

$$\|\nabla u_\lambda\|_2^2 + \int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 dx < C.$$

**Theorem 1.2.** For  $\lambda = 0$ , let  $\alpha \in [0, 8)$  and  $\max\{2, \frac{2+\alpha}{2}\} < p < 5$ . Then, problem (1.1) has at least a positive solution  $u \in E \cap C^2(\mathbb{R}^3 \setminus T)$  in the sense of (1.12).

## 2 Bounded nonnegative (PS) sequence

In this section,  $\lambda > 0$  is always assumed. Our aim is to know how the functional  $I_\lambda$  defined in (1.9) has always a bounded nonnegative (PS) sequence at some level  $c > 0$  in  $H$ . As mentioned in the introduction, the authors in [6] developed an approach to get a bounded (PS) sequence for the single equation (1.14) with certain nonlinearities. By improving some techniques used in [6], we are able to obtain a bounded nonnegative (PS) sequence for (1.1), the nonnegativity of the (PS) sequence helps us to estimate the related term caused by the nonlocal term  $\phi(x)u$ , which leads to a nonzero weak limit of the (PS) sequence. Let us recall first a deformation lemma from [24].

**Lemma 2.1.** ([24], Lemma 2.3) Let  $X$  be a Banach space,  $\varphi \in C^1(X, \mathbb{R})$ ,  $S \subset X$ ,  $c \in \mathbb{R}$ ,  $\varepsilon, \delta > 0$  such that for any  $u \in \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$ :  $\varphi'(u) \geq 8\varepsilon/\delta$ . Then there exists  $\eta \in C([0, 1] \times X, X)$  such that

- (i)  $\eta(t, u) = u$ , if  $t = 0$  or  $u \notin \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$ .
- (ii)  $\eta(1, \varphi^{c+\varepsilon} \cap S) \subset \varphi^{c-\varepsilon}$ , where  $\varphi^{c \pm \varepsilon} = \{u \in X : \varphi(u) \leq c \pm \varepsilon\}$ .
- (iii)  $\eta(t, \cdot)$  is an homeomorphism of  $X$ , for any  $t \in [0, 1]$ .

(iv)  $\varphi(\eta(\cdot, u))$  is non increasing, for any  $u \in X$ .

Now, we give some lemmas, by which Lemma 2.1 can be used to get a desirable (PS) sequence.

**Lemma 2.2.** *Let  $M > 0$  be a constant. If  $u_1, u_2 \in H$  with  $\lambda > 0$  and  $\|u_1\|_H, \|u_2\|_H \leq M$ , then there exist  $C := C(M, p) > 0$  such that*

$$\|I'(u_1) - I'(u_2)\|_{H'} \leq C (\|u_1 - u_2\|_H + \|u_1 - u_2\|_H^3). \quad (2.1)$$

*Proof.* By (1.10) and (1.6),

$$\begin{aligned} \langle I'(u_1) - I'(u_2), \psi \rangle_H &= \langle u_1 - u_2, \psi \rangle_H + \int_{\mathbb{R}^3} (\phi_{u_1} u_1 - \phi_{u_2} u_2) \psi dx \\ &\quad - \int_{\mathbb{R}^3} (|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2) \psi dx, \end{aligned}$$

hence (2.1) is proved if we have that

$$\left| \int_{\mathbb{R}^3} (|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2) \psi dx \right| \leq C \|u_1 - u_2\|_H \|\psi\|_H, \quad (2.2)$$

$$\int_{\mathbb{R}^3} (\phi_{u_1} u_1 - \phi_{u_2} u_2) \psi dx \leq C (\|u_1 - u_2\|_H + \|u_1 - u_2\|_H^3) \|\psi\|_H. \quad (2.3)$$

Indeed, using Taylor's formula and Hölder inequality as well as Minkovski inequality, we see that there is a function  $\theta$  with  $0 < \theta < 1$  such that

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2) \psi dx \right| &\leq p \|u_1 - u_2\|_{p+1} \|\psi\|_{p+1} \|(\theta u_1 + (1-\theta)u_2)\|_{p+1}^{p+1} \\ &\leq p (\|u_1\|_{p+1} + \|u_2\|_{p+1})^{p+1} \|u_1 - u_2\|_{p+1} \|\psi\|_{p+1} \leq p(2M)^{p+1} \|u_1 - u_2\|_{p+1} \|\psi\|_{p+1}, \end{aligned}$$

hence (2.2) is obtained. To prove (2.3), we let  $v = u_2 - u_1$ , it follows from (1.7) that

$$\int_{\mathbb{R}^3} (\phi_{u_2} u_2 - \phi_{u_1} u_1) \psi dx = J_1 + J_2 + J_3 + J_4 + J_5, \quad (2.4)$$

where

$$J_1 = \int_{\mathbb{R}^3} \frac{v^2(y) v(x) \psi(x)}{|x-y|} dx dy \leq C \|v\|_H^3 \|\psi\|_H$$

$$J_2 = \int_{\mathbb{R}^3} \frac{v^2(y) u_1(x) \psi(x)}{|x-y|} dx dy \leq C \|v\|_H^2 \|u_1\|_H \|\psi\|_H$$

$$J_3 = \int_{\mathbb{R}^3} \frac{u_1^2(y) v(x) \psi(x)}{|x-y|} dx dy \leq C \|u_1\|_H^2 \|v\|_H \|\psi\|_H$$

$$J_4 = 2 \int_{\mathbb{R}^3} \frac{u_1(y) u_1(x) v(y) \psi(x)}{|x-y|} dx dy \leq C \|u_1\|_H^2 \|v\|_H \|\psi\|_H$$

$$J_5 = 2 \int_{\mathbb{R}^3} \frac{u_1(y) v(y) v(x) \psi(x)}{|x-y|} dx dy \leq C \|u_1\|_H \|v\|_H^2 \|\psi\|_H$$

here we used the following Hardy-Littlewood-Sobolev inequality (Theorem 4.3 of [19])

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x) |x-y|^{-\lambda} h(y) dx dy \right| \leq C(n, \lambda, p) \|f\|_p \|h\|_r,$$

where  $p, r > 1$  and  $0 < \lambda < N$  with  $\frac{1}{p} + \frac{\lambda}{N} + \frac{1}{r} = 2$ ,  $f \in L^p(\mathbb{R}^N)$  and  $h \in L^r(\mathbb{R}^N)$ , the sharp constant  $C(N, \lambda, p)$ , independent of  $f$  and  $h$ . Then (2.3) holds by (2.4) and the estimates for  $J_1$  to  $J_5$ . Hence Lemma 2.2 is proved.  $\square$

Before giving our next lemma, we recall some basic properties of  $\phi_u(x)$  given by (1.7). Let

$$u_t := u_t(x) = t^2 u(tx) \text{ for } t > 0 \text{ and } x \in \mathbb{R}^3,$$

then  $u(x) = (u_t)_{\frac{1}{t}}(x) = (u_{\frac{1}{t}})_t(x)$  and

$$\|\nabla u_t\|_2^2 = t^3 \|\nabla u\|_2^2, \quad \|u_t\|_p^p = t^{2p-3} \|u\|_p^p \text{ for } 1 \leq p < \infty, \quad (2.5)$$

$$\int_{\mathbb{R}^3} \phi_{u_t} u_t^2 dx = t^3 \int_{\mathbb{R}^3} \phi_u u^2 dx, \quad \int_{\mathbb{R}^3} \frac{u_t^2}{|y|^\alpha} dx = t^{1+\alpha} \int_{\mathbb{R}^3} \frac{u^2}{|y|^\alpha} dx. \quad (2.6)$$

**Lemma 2.3.** *If  $\alpha \in [0, 8)$  and  $\max\{2, \frac{\alpha+2}{2}\} < p < 5$ , then there exist  $\rho > 0$ ,  $\delta > 0$ ,  $e \in H$  with  $e \geq 0$  and  $\|e\|_H > \rho$  such that*

(i)  $I(u) \geq \delta$ , for all  $u \in H$  with  $\|u\|_H = \rho$ .

(ii)  $I(e) < I(0) = 0$ .

*Proof.* (i) Since  $H \hookrightarrow L^p(\mathbb{R}^3)$  for  $2 \leq p < 6$ , this conclusion is a straightforward consequence of the definition of  $I$ .

(ii) For  $t > 0$  and  $u \in H \setminus \{0\}$ , by (2.5), (2.6) and the definition of  $I$ , we see that

$$I(u_t) = \frac{t^3}{2} \|\nabla u\|_2^2 + \frac{\lambda t}{2} \|u\|_2^2 + \frac{t^{1+\alpha}}{2} \int_{\mathbb{R}^3} \frac{u^2}{|y|^\alpha} dx + \frac{t^3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 dx - \frac{t^{2p-1}}{p+1} \|u\|_{p+1}^{p+1}. \quad (2.7)$$

Since  $\alpha \in [0, 8)$ ,  $p > \max\{2, \frac{\alpha+2}{2}\}$ , we see  $I(u_t) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . Hence, for each  $u \in H \setminus \{0\}$ , there is a  $t_* > 0$  large enough such that (ii) holds with  $e = u_{t_*}$ . Moreover, we may assume that  $e \geq 0$ , otherwise, just replace  $e$  by  $|e|$ .  $\square$

For each  $\lambda > 0$  and  $e$  given by Lemma 2.3, define

$$c := c_\lambda = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I_\lambda(u), \quad (2.8)$$

where  $\Gamma := \{\gamma \in C([0,1]; H) : \gamma(0) = 0, \gamma(1) = e\}$ . Clearly,  $c > 0$  by lemma 2.3. Let  $\{t_n\} \subset (0, +\infty)$  be a sequence such that  $t_n \rightarrow 1$  as  $n \rightarrow +\infty$ , then by (2.5) it is easy to show that

$$e_{t_n} := t_n^2 e(t_n x) \rightarrow e \text{ in } H, \text{ as } n \rightarrow +\infty. \quad (2.9)$$

Since  $I \in C^1(H)$ , it follows from Lemma 2.3 (ii) that there is  $\varepsilon > 0$  small enough such that  $I(u) < 0$  for all  $u \in B_\varepsilon(e)$ . Again using (2.9), there exists  $t_0 \in (0, 1)$  such that

$$e_t := t^2 e(tx) \in B_\varepsilon(e) \text{ for all } t \in (t_0, 1). \quad (2.10)$$

For this  $t_0 \in (0, 1)$ , similar to [6] we have

**Lemma 2.4.** *Let  $t_0$  be given by (2.10). Then for all  $t \in (t_0, 1)$ , we have*

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u_t)$$

where  $c$  and  $\Gamma$  are defined in (2.8),  $u_t = t^2 u(tx)$ .

*Proof.* The proof is the same as that of Lemma 11 in [6].  $\square$

By Lemma 2.4, we know that for any  $s \in (t_0, 1)$  there exists  $\gamma_s \in \Gamma$  such that

$$\max_{u \in \gamma_s([0,1])} I(u_s) \leq c + (1 - s^3). \quad (2.11)$$

For  $s \in (t_0, 1)$ , we define a set

$$U_s := \{u \in \gamma_s([0,1]) : I(u) \geq c - (1 - s^3)\}, \quad (2.12)$$

then, (2.8) and the definition of  $U_s$  imply that  $U_s \neq \emptyset$  for  $s \in (t_0, 1)$ .

**Lemma 2.5.** *If  $\alpha \in [0, 8)$  and  $\max\{2, \frac{\alpha+2}{2}\} < p < 5$ , then for  $t_0$  given by (2.10) there exist  $t^* \in (t_0, 1)$  and  $M = \frac{2(c+2)(2p-1)}{(p-2)t^{*3}} + \frac{4(c+2)(2p-1)}{(2p-2-\alpha)t^{*1+\alpha}}$  such that*

$$\|u\|_H^2 + \int_{\mathbb{R}^3} \phi_u u^2 dx < M \quad \text{for all } u \in U_s \text{ with } s \in (t^*, 1).$$

**Proof:** Let  $u \in U_s$  and note that  $u(x) = (u_s)_\perp$ , it follows from (2.5), (2.6) and the definition (1.9) that

$$\begin{aligned} I(u_s) - I(u) &= \frac{1}{2}(1 - \frac{1}{s^3}) \|\nabla u_s\|_2^2 + \frac{\lambda}{2}(1 - \frac{1}{s}) \|u_s\|_2^2 + \frac{1}{2}(1 - \frac{1}{s^{1+\alpha}}) \int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha} dx \\ &\quad + \frac{1}{4}(1 - \frac{1}{s^3}) \int_{\mathbb{R}^3} \phi_{u_s} u_s^2 dx - \frac{1}{p+1}(1 - \frac{1}{s^{2p-1}}) \|u_s\|_{p+1}^{p+1}. \end{aligned} \quad (2.13)$$

For  $u \in U_s$ , (2.11) and (2.12) implies that

$$I(u_s) - I(u) \leq 2(1 - s^3), \quad \text{for } s \in (t_0, 1). \quad (2.14)$$

By calculation, this and (2.13) show that, for any  $u \in U_s$ ,

$$\begin{aligned} \frac{\lambda}{2} \frac{s^2 - s^3}{s^3 - 1} \|u_s\|_2^2 &+ \frac{1}{2} \frac{s^2 - s^{3+\alpha}}{s^{3+\alpha} - s^\alpha} \int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha} dx + \frac{1}{p+1} \frac{s^{2p+2} - s^3}{s^{2p+2} - s^{2p-1}} \|u_s\|_{p+1}^{p+1} \\ &- \frac{1}{2} \|\nabla u_s\|_2^2 - \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_s} u_s^2 dx \leq 2s^3. \end{aligned} \quad (2.15)$$

To simplify (2.15), we need to use the following facts:

$$\frac{s^2 - s^3}{s^3 - 1} = \frac{s^2}{s^2 + s + 1} \geq -1 \quad \text{for } s \geq 0.$$

$$g(s) \triangleq \frac{s^2 - s^{3+\alpha}}{s^{3+\alpha} - s^\alpha} = \frac{s^{2-\alpha} - s^3}{s^3 - 1} \xrightarrow{s \rightarrow 1^-} -\frac{1+\alpha}{3}, \quad \text{and } g(s) \equiv g(1) = -1 \text{ if } \alpha = 2.$$

$p > \frac{\alpha+2}{2}$  implies that  $\frac{2p-1}{1+\alpha} > 1$  and  $\varepsilon_0 := \frac{2p+\alpha}{2(1+\alpha)} \in (1, \frac{2p-1}{1+\alpha})$ . Hence, there is  $\delta_1 > 0$  small enough such that  $1 - \delta_1 > t_0$  and

$$g(s) \geq -\frac{\varepsilon_0(1+\alpha)}{3} = \frac{2p+\alpha}{6} \quad \text{for all } s \in (1 - \delta_1, 1).$$

Let

$$h(s) = \frac{s^{2p+2} - s^3}{s^{2p+2} - s^{2p-1}} = \frac{s^3 - s^{4-2p}}{s^3 - 1} \quad \text{for } s \in (0, 1),$$

then

$$\lim_{s \rightarrow 1^-} h(s) = \frac{2p-1}{3} \quad \text{and} \quad h'(s) = \frac{(2p-1)s^{6-2p} - 3s^2 - (2p-4)s^{3-2p}}{(s^3 - 1)^2},$$

$$h'(s) \xrightarrow{s \rightarrow 1^-} -\frac{(2p-1)(p-2)}{3} < 0 \quad \text{if } p > 2.$$

This shows that there is  $\delta_2 > 0$  small enough and  $1 - \delta_2 > t_0$  such that

$$h'(s) < 0 \text{ and } h(s) \geq \lim_{s \rightarrow 1^-} h(s) = \frac{2p-1}{3} \text{ for all } s \in (1 - \delta_2, 1) \text{ and } p > 2.$$

For  $p = 2$ ,  $h(s) \equiv \frac{2p-1}{3} = 1$ , so we see that

$$h(s) \geq \frac{2p-1}{3} \text{ for all } s \in (1 - \delta_2, 1) \text{ and } p \geq 2.$$

So, for  $s \in (t^*, 1)$  with  $t^* = 1 - \min\{\delta_1, \delta_2\}$ , it follows from (2.15) that

$$\begin{aligned} -\frac{\lambda}{2}\|u_s\|_2^2 &= \frac{2p+\alpha}{12} \int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha} dx + \frac{1}{p+1} \frac{2p-1}{3} \|u_s\|_{p+1}^{p+1} \\ &= \frac{1}{2} \|\nabla u_s\|_2^2 - \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_s} u_s^2 dx \leq 2s^3. \end{aligned}$$

That is

$$\begin{aligned} -\frac{1}{p+1} \|u_s\|_{p+1}^{p+1} &\geq -\frac{3}{2p-1} \left( \frac{\lambda}{2} \|u_s\|_2^2 + \frac{1}{2} \|\nabla u_s\|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_s} u_s^2 dx \right) \\ &\quad - \frac{2p+\alpha}{4(2p-1)} \int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha} dx - \frac{6}{2p-1} s^3. \end{aligned} \quad (2.16)$$

For  $u \in U_s$ , by (2.11) it gives that

$$\begin{aligned} &\left( \frac{\lambda}{2} \|u_s\|_2^2 + \frac{1}{2} \|\nabla u_s\|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_s} u_s^2 dx \right) \\ &+ \frac{1}{2} \int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha} dx - \frac{1}{p+1} \|u_s\|_{p+1}^{p+1} \leq c + (1 - s^3). \end{aligned} \quad (2.17)$$

Hence, it follows from (2.16) and (2.17) that

$$\begin{aligned} &\frac{2p-2-\alpha}{4(2p-1)} \int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha} dx + \frac{2p-4}{2p-1} \left( \frac{\lambda}{2} \|u_s\|_2^2 + \frac{1}{2} \|\nabla u_s\|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_s} u_s^2 dx \right) \\ &\leq c + 1 - \frac{2p-7}{2p-1} s^3 \\ &\leq c + 1 + \left| \frac{2p-7}{2p-1} \right| \leq c + 2 \text{ if } p > 2 \text{ and } s < 1. \end{aligned}$$

This implies that, if  $5 > p > \max\{2, \frac{\alpha+2}{2}\}$  and  $s \in (t^*, 1)$

$$\frac{\lambda}{2} \|u_s\|_2^2 + \frac{1}{2} \|\nabla u_s\|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_s} u_s^2 dx \leq \frac{(c+2)(2p-1)}{2(p-2)}. \quad (2.18)$$

and

$$\frac{1}{4} \int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha} dx \leq \frac{(c+2)(2p-1)}{2p-2-\alpha} \text{ for } \alpha \in [0, 8).$$

Hence, it follows from (2.18) and by using (2.5) and (2.6)

$$\frac{\lambda}{2} s \|u\|_2^2 + \frac{1}{2} s^3 \|\nabla u\|_2^2 + \frac{1}{4} s^3 \int_{\mathbb{R}^3} \phi_u u^2 dx \leq \frac{(c+2)(2p-1)}{2(p-2)}.$$

Since  $s \in (t^*, 1)$ ,  $s \geq s^3 \geq t^{*3}$  and  $s^{1+\alpha} \geq t^{*1+\alpha}$  for  $\alpha \in [0, 8)$ , those and  $p > \max\{2, \frac{\alpha+2}{2}\}$  imply that

$$\|u\|_H^2 + \int_{\mathbb{R}^3} \phi_u u^2 dx \leq \frac{2(c+2)(2p-1)}{(p-2)t^{*3}} + \frac{4(c+2)(2p-1)}{(2p-2-\alpha)t^{*1+\alpha}}, \quad (2.19)$$



and Lemma 2.5 is proved by taking  $M = \frac{2(c+2)(2p-1)}{(p-2)t^{*3}} + \frac{4(c+2)(2p-1)}{(2p-2-\alpha)t^{*1+\alpha}}$ .  $\square$

Note that  $M$  given by the above lemma depends on  $\lambda$ , since  $c$  depends on  $\lambda$  by the definition of  $I$ . The following lemma is for getting a bounded (PS) sequence. In this lemma, the constant  $M$  can be chosen independent of  $\lambda$  if  $\lambda \in (0, 1]$ .

**Lemma 2.6.** *If  $\alpha \in [0, 8)$ ,  $\max\{2, \frac{\alpha+2}{2}\} < p < 5$  and  $c$  be given by (2.8). Then there exists a bounded nonnegative sequence  $\{u_n\} \subset H$  such that*

$$I(u_n) \rightarrow c > 0, \quad I'(u_n) \rightarrow 0 \text{ as } n \rightarrow +\infty, \quad (2.20)$$

Moreover, if  $\lambda \in (0, 1]$  there exists  $M > 0$  which is independent of  $\lambda \in (0, 1]$  such that

$$\|u_n\|_H^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \leq M.$$

**Proof:** For  $t \in (t^*, 1)$  with  $t^*$  given in Lemma 2.5, let

$$W_t = \{u : u \in U_t\}, \quad U_t \text{ defined in (2.12)}, \quad (2.21)$$

and then for  $u \in W_t$ , by (2.19) (2.7) and (2.11) we have that

$$\begin{aligned} I(u) - I(u_t) &= \frac{1}{2}(1-t^3)\|\nabla u\|_2^2 + \frac{\lambda}{2}(1-t)\|u\|_2^2 + \frac{1-t^{1+\alpha}}{2} \int_{\mathbb{R}^3} \frac{u^2}{|y|^\alpha} dx \\ &\quad + \frac{1-t^3}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{1-t^{2p-1}}{p+1} \|u\|_{p+1}^{p+1} \\ &\leq (1-t^3) \left( \frac{\lambda}{2} \|u\|_2^2 + \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx \right) \\ &\quad + \frac{1-t^{2p-1}}{t^{2p-1}} \left( \frac{1}{2} \int_{\mathbb{R}^3} \frac{u_t^2}{|y|^\alpha} dx - \frac{1}{p+1} \|u_t\|_{p+1}^{p+1} \right) \\ &\leq (1-t^3) \left( \frac{\lambda}{2} \|u\|_2^2 + \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx \right) + \frac{1-t^{2p-1}}{t^{2p-1}} I(u_t) \\ &\leq (1-t^3) \frac{(c+2)(2p-1)}{(2p-4)t^{*3}} + (1-t^{2p-1}) \frac{c+1}{t^{*2p-1}} \rightarrow 0 \text{ as } t \rightarrow 1^-. \end{aligned}$$

On the other hand, similar to (2.14) we know that

$$I(u_t) - I(u) \leq 2(1-t^3) \rightarrow 0 \text{ as } t \rightarrow 1^-.$$

Hence,

$$\limsup_{t \rightarrow 1^-} \sup_{u \in W_t} |I(u_t) - I(u)| = 0. \quad (2.22)$$

Define

$$S = \left\{ |u| : u \in H \text{ and } \|u\|_H^2 + \int_{\mathbb{R}^3} \phi_u u^2 dx < M \right\}, \text{ where } M \text{ is given by Lemma 2.5,}$$

$$S_\delta = \{u : u \in H \text{ and } \text{dist}(u, S) < \delta\}, \quad \delta \in (0, 1). \quad (2.23)$$

Clearly,  $\|v\|_H \leq \sqrt{M} + 1$  for all  $v \in S_\delta$ . Then, by Lemma 2.2, there is a constant  $K := K(M)$  such that

$$\|I'(u) - I'(v)\|_{H'} \leq K \|u - v\|_H \text{ for all } u, v \in S_\delta. \quad (2.24)$$

and since  $I \in C^1(H, \mathbb{R})$ , there exists  $C_S > 0$  such that

$$\|I(u) - I(v)\|_H \leq C_S \|u - v\|_H \text{ for all } u, v \in S_\delta. \quad (2.25)$$

For any  $m \in \mathbb{N}$  and  $M$  given by Lemma 2.5, let

$$\Lambda_m = \left\{ |u| : u \in H, \|u\|_H^2 + \int_{\mathbb{R}^3} \phi_u u^2 dx < M + \frac{1}{m} \text{ and } |I(u) - c| \leq \frac{C_S + 1}{\sqrt{m}} \right\}. \quad (2.26)$$

We claim that  $\Lambda_m \neq \emptyset$ . Indeed, for any  $m \geq 1$ , since (2.22) we can find  $t_m \in (t^*, 1)$  such that

$$1 - t_m^3 < \frac{1}{32m} \text{ and } I(u) \leq I(u_{t_m}) + \frac{1}{32m} \text{ for all } u \in W_{t_m}.$$

Then it follows from (2.11) and (2.12) that

$$c - \frac{1}{32m} \leq I(u) \leq c + \frac{1}{16m} \text{ for all } u \in W_{t_m}. \quad (2.27)$$

By the definition of  $W_{t_m}$ , Lemma 2.5 implies that  $\|u\|_H^2 + \int_{\mathbb{R}^3} \phi_u u^2 dx \leq M$  for all  $u \in W_{t_m}$ . This and (2.27) show that  $W_{t_m} \subset \Lambda_m$ , that is  $\Lambda_m \neq \emptyset$ .

Next, we claim that there are infinitely many elements in  $\{\Lambda_m\}_{m=1}^{+\infty}$ , which we still simply denote by  $\Lambda_m$  ( $m = 1, 2, \dots$ ), such that for each  $m \geq 1$ , there is  $u_m \in \Lambda_m$  with

$$\|I'(u_m)\|_{H'} < \frac{1+K}{\sqrt{m}}, \quad K \text{ is given by (2.24)}. \quad (2.28)$$

Then, to prove Lemma 2.6 we need only to show the above claim. By contradiction, if the claim is false, then there must be a number  $\bar{m} \in \mathbb{N}$  with  $\bar{m} > \max\{\frac{1}{8c}, 4\}$  such that

$$\|I'(u)\|_{H'} \geq \frac{1+K}{\sqrt{m}}, \text{ for all } m > \bar{m} \text{ and } u \in \Lambda_m. \quad (2.29)$$

By the above discussion we know that  $W_{t_m} \subset \Lambda_m$ . For any  $u \in W_{t_m}$ , the definition of  $W_{t_m}$  and Lemma 2.5 show that  $\|u\|_H^2 + \int_{\mathbb{R}^3} \phi_u u^2 dx \leq M$  and  $W_{t_m} \subset S$ . Hence,

$$W_{t_m} \subset S \cap \{u \in H : |I(u) - c| < \frac{1}{8m}\} \subset S \cap \{u \in H : |I(u) - c| < \frac{C_S + 1}{\sqrt{m}}\} \subset \Lambda_m,$$

where (2.27) is used. Then

$$S \cap \{u \in H : |I(u) - c| < \frac{C_S + 1}{\sqrt{m}}\} \neq \emptyset.$$

Let  $\varepsilon = \frac{1}{16m}$ ,  $\delta = \frac{1}{2\sqrt{m}}$ , then  $\frac{8\varepsilon}{\delta} = \frac{1}{\sqrt{m}} < \frac{1}{2} < 1$ , since  $\bar{m} > \max\{\frac{1}{8c}, 4\}$ . So,

$$(S)_{2\delta} = S_{\frac{1}{\sqrt{m}}} = \left\{ u : u \in H \text{ and } \text{dist}(u, S) < \frac{1}{\sqrt{m}} \right\}.$$

By the definitions of  $S$  and  $\Lambda_m$ , we have

$$S \cap \{u \in H : |I(u) - c| < \frac{C_S + 1}{\sqrt{m}}\} \subset \Lambda_m.$$

Hence, for any  $u \in S \cap \{u \in H : |I(u) - c| < \frac{C_S + 1}{\sqrt{m}}\} \subset \Lambda_m$ ,

$$\|I'(u)\|_{H'} \geq \frac{1+K}{\sqrt{m}}, \text{ for all } m > \bar{m}. \quad (2.30)$$

For any  $v \in S_{\frac{1}{\sqrt{m}}} \cap \{u \in H : |I(u) - c| < \frac{1}{8m}\}$ , it is not difficult to know that there is  $u_0 \in S$  such that

$$\|u_0 - v\|_H < \frac{1}{\sqrt{m}}. \quad (2.31)$$

This and (2.25) show that

$$\begin{aligned}
\|I(u_0) - c\|_H &\leq \|I(v) - I(u_0)\|_H + \|I(v) - c\|_H \\
&\leq \|I(v) - c\|_H + \frac{C_S}{\sqrt{m}} \\
&\leq \frac{1}{8m} + \frac{C_S}{\sqrt{m}} \leq \frac{C_S + 1}{\sqrt{m}}.
\end{aligned}$$

That is  $u_0 \in S \cap \{u \in H : |I(u) - c| < \frac{C_S + 1}{\sqrt{m}}\}$ . Then, it follows from (2.24), (2.30) and (2.31) that, for  $v \in S_{\frac{1}{\sqrt{m}}} \cap \{u \in H : |I(u) - c| < \frac{1}{8m}\}$ ,

$$\begin{aligned}
\|I'(v)\|_{H'} &= \|I'(v) - I'(u_0) + I'(u_0)\|_{H'} \\
&\geq \|I'(u_0)\|_{H'} - \|I'(v) - I'(u_0)\|_{H'} \\
&\geq \frac{1 + K}{\sqrt{m}} - K\|u_0 - v\|_H \\
&\geq \frac{1 + K}{\sqrt{m}} - K\frac{1}{\sqrt{m}} = \frac{1}{\sqrt{m}}.
\end{aligned}$$

Applying Lemma 2.1 with  $X = H$ ,  $\varphi = I$ , we know that there is an homeomorphism  $\eta(t, \cdot) : [0, 1] \times H \rightarrow H$  such that

$$\eta(t, u) = u, \text{ if } t = 0 \text{ or } u \notin S_{\frac{1}{\sqrt{m}}} \cap \{u \in H : |I(u) - c| \leq \frac{1}{8m}\}; \quad (2.32)$$

$$I(\eta(1, u)) \leq c - \frac{1}{16m}, \text{ for } u \in S \cap \{u \in H : |I(u) - c| \leq \frac{1}{8m}\}; \quad (2.33)$$

$$I(\eta(t, u)) \leq I(u), \text{ for any } u \in H. \quad (2.34)$$

Let  $\xi(u) := \eta(1, u)$  and  $\bar{\gamma}(t) = \xi(|\gamma_{t_m}(t)|) \in C([0, 1], H)$ . By  $m > \bar{m} > \max\{\frac{1}{8c}, 4\}$ ,  $c > \frac{1}{8m}$ , then  $\{0, e\} \not\subset S_{\frac{1}{\sqrt{m}}} \cap \{u \in H : |I(u) - c| < \frac{1}{8m}\}$ , since  $I(e) < 0$  and  $|I(e) - c| = c + |I(e)| > c$  where  $e$  is given by Lemma 2.3. With this observation and (2.32) we see that  $\bar{\gamma}(0) = \xi(|\gamma_{t_m}(0)|) = \xi(0) = \eta(1, 0) = 0$ ,  $\bar{\gamma}(1) = \xi(|\gamma_{t_m}(1)|) = \xi(e) = \eta(1, e) = e$ . Hence,  $\bar{\gamma} \in \Gamma$ , with  $\Gamma$  defined in (2.8). For each  $m \geq \bar{m}$ , let  $u_m \in \bar{\gamma}([0, 1])$  be such that

$$I(\xi(|u_m|)) = \max_{u \in \gamma_{t_m}[0, 1]} I(\xi(|u|)) = \max_{v \in \bar{\gamma}[0, 1]} I(v) \geq c. \quad (2.35)$$

Since  $u_m \in \gamma_{t_m}[0, 1]$ ,  $|u_m| \in |\gamma_{t_m}[0, 1]| = \{|u| : u \in \gamma_{t_m}[0, 1]\}$ . We are ready to get a contradiction in both of the following two cases.

**Case A:** If  $|u_m| \in |\gamma_{t_m}[0, 1]| \setminus U_{t_m}$ , then (2.34) and the definition of  $U_{t_m}$  imply that

$$I(\xi(|u_m|)) = I(\eta(1, |u_m|)) \leq I(u_m) \leq c - (1 - t_m^3) < c,$$

which contradicts (2.35).

**Case B:** If  $|u_m| \in U_{t_m}$ , then by (2.21)  $|u_m| \in W_{t_m}$  and (2.27) implies that  $|I(|u_m|) - c| \leq \frac{1}{16m}$ . Moreover,  $\|u_m\|_H^2 + \int_{\mathbb{R}^3} \phi_{u_m} u_m^2 dx \leq M$  by Lemma 2.5. Hence  $|u_m| \in S \cap \{u \in H : |I(u) - c| \leq \frac{1}{16m}\}$ , and it follows from (2.33) that

$$I(\xi(|u_m|)) = I(\eta(1, |u_m|)) \leq c - \frac{1}{16m} < c,$$

this is a contradiction to (2.35).  $\square$

### 3 Existence for $\lambda > 0$ : Proof of Theorem 1.1.

Motivated by [5], we prove Theorem 1.1 by a result due to S.Solimini [22], which is a version of so called concentration-compactness principle. To state this result, we should recall the operator  $T_{s, \xi}$  and its basic properties. Let  $s > 0$ ,  $N \geq 3$  and  $\xi \in \mathbb{R}^N$  be fixed, for any  $u \in L^q(\mathbb{R}^N)$  ( $1 < q < +\infty$ ) we define

$$T_{s, \xi} u(x) \triangleq T(s, \xi) u(x) := s^{-\frac{N-2}{2}} u(s^{-1}x + \xi), \quad \forall x \in \mathbb{R}^N. \quad (3.1)$$

Clearly,  $T(s, \xi)u \in L^q(\mathbb{R}^N)$  if  $u \in L^q(\mathbb{R}^N)$  and  $T(s, \xi)$  is also well defined on Hilbert space  $D^{1,2}(\mathbb{R}^N)$  with scalar product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \nabla u \nabla v dx, \text{ for } u, v \in D^{1,2}(\mathbb{R}^N), \quad (3.2)$$

since  $T(s, \xi)u \in D^{1,2}(\mathbb{R}^N)$  if  $u \in D^{1,2}(\mathbb{R}^N)$ . It is not difficult to see that the linear operators

$$u \in L^{2^*}(\mathbb{R}^N) \mapsto T(s, \xi)u \in L^{2^*}(\mathbb{R}^N) \text{ and } u \in D^{1,2}(\mathbb{R}^N) \mapsto T(s, \xi)u \in D^{1,2}(\mathbb{R}^N)$$

are isometric, where  $2^* = \frac{2N}{N-2}$ . Moreover, we have that

$$T_{s,\xi}^{-1} = T(s^{-1}, -s\xi), \quad T_{s,\xi}T_{\mu,\eta} = T(s\mu, \xi/\mu + \eta). \quad (3.3)$$

$$\|\nabla T_{s,\xi}u\|_2^2 = \|\nabla u\|_2^2, \quad \|T_{s,\xi}u\|_q^q = s^{N-\frac{q(N-2)}{2}}\|u\|_q^q. \quad (3.4)$$

For  $N \geq 3$ ,  $k \in [2, N)$  and  $x \in \mathbb{R}^N$ , in this section we denote that

$$x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \text{ i.e. } y \in \mathbb{R}^k, z \in \mathbb{R}^{N-k},$$

$\tilde{y} = (y, 0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ ,  $\tilde{z} = (0, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ . Similarly,  $x_n = (y_n, z_n) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ ,  $\tilde{y}_n = (y_n, 0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ .

**Lemma 3.1.** ([5], Proposition 22) Let  $\{\eta_n\} \subset \mathbb{R}^N$  be such that  $\lim_{n \rightarrow \infty} |\eta_n| = \infty$  and fix  $R > 0$ . Then for any  $m \in \mathbb{N} \setminus \{0, 1\}$  there exists  $N_m \in \mathbb{N}$  such that for any  $n > N_m$  one can find a sequence of unit orthogonal matrices,  $\{g_i\}_{i=1}^m \in O(N)$  satisfying the condition

$$B_R(g_i\eta_n) \cap B_R(g_j\eta_n) = \emptyset, \quad \text{for } i \neq j.$$

**Lemma 3.2.** ([5], Proposition 11) Let  $q \in (1, \infty)$  and  $\{s_n\} \subset (0, \infty)$ ,  $\{\xi_n\} \subset \mathbb{R}^N$  be such that  $s_n \xrightarrow{n} s \neq 0$ ,  $\xi_n \xrightarrow{n} \xi$ . Then

$$T_{s_n, \xi_n} u_n \xrightarrow{n} T_{s, \xi} u \text{ weakly in } L^q(\mathbb{R}^N),$$

if  $u_n \xrightarrow{n} u$  weakly in  $L^q(\mathbb{R}^N)$ .

**Lemma 3.3.** Let  $\{s_n\} \subset (0, \infty)$ ,  $\{\xi_n\} \subset \mathbb{R}^N$  be such that  $s_n \xrightarrow{n} s_0 \neq 0$ ,  $\xi_n \xrightarrow{n} \xi$ . If  $v_n \xrightarrow{n} v$  weakly in  $D^{1,2}(\mathbb{R}^N)$ , then

$$T_{s_n, \xi_n} v_n \xrightarrow{n} T_{s_0, \xi} v \text{ weakly in } D^{1,2}(\mathbb{R}^N).$$

*Proof.* For any  $\varphi \in C_0^\infty(\mathbb{R}^N)$ , by (3.2) we get that

$$\langle T_{s_n, 0}^{-1} v_n, \varphi \rangle = \langle v_n, T_{s_n, 0} \varphi \rangle = \langle v_n, T_{s_0, 0} \varphi \rangle + \langle v_n, T_{s_n, 0} \varphi - T_{s_0, 0} \varphi \rangle. \quad (3.5)$$

Since

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\nabla(T_{s_n, 0} \varphi - T_{s_0, 0} \varphi)\|_2^2 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla T_{s_n, 0} \varphi|^2 dx + \int_{\mathbb{R}^N} |\nabla T_{s_0, 0} \varphi|^2 dx \\ &\quad - 2 \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \nabla T_{s_n, 0} \varphi \nabla T_{s_0, 0} \varphi = 0, \end{aligned}$$

we have

$$\langle v_n, T_{s_n, 0} \varphi - T_{s_0, 0} \varphi \rangle \leq \|\nabla v_n\|_2 \|\nabla(T_{s_n, 0} \varphi - T_{s_0, 0} \varphi)\|_2 \xrightarrow{n} 0. \quad (3.6)$$

By  $T_{s_0, 0} \varphi \in C_0^\infty(\mathbb{R}^N)$  and  $v_n \xrightarrow{n} v$  weakly in  $D^{1,2}(\mathbb{R}^N)$ , we have

$$\langle v_n, T_{s_0, 0} \varphi \rangle \xrightarrow{n} \langle v, T_{s_0, 0} \varphi \rangle = \langle T_{s_0, 0}^{-1} v, \varphi \rangle. \quad (3.7)$$

It follows from (3.5) to (3.7) that

$$\langle T_{s_n,0}^{-1}v_n, \varphi \rangle \xrightarrow{n} \langle T_{s_0,0}^{-1}v, \varphi \rangle, \text{ for any } \varphi \in C_0^\infty(\mathbb{R}^N). \quad (3.8)$$

On the other hand, for any  $\psi \in D^{1,2}(\mathbb{R}^N)$  and any  $\epsilon > 0$ , there exists  $\varphi \in C_0^\infty(\mathbb{R}^N)$  such that  $\|\nabla(\psi - \varphi)\|_2 < \epsilon$  and

$$\langle T_{s_n,0}^{-1}v_n, \psi - \varphi \rangle \leq \|\nabla(T_{s_n,0}^{-1}v_n)\|_2 \|\nabla(\psi - \varphi)\|_2 = \|\nabla v_n\|_2 \|\nabla(\psi - \varphi)\|_2,$$

this and (3.8) imply that

$$\langle T_{s_n,0}^{-1}v_n, \varphi \rangle \xrightarrow{n} \langle T_{s_0,0}^{-1}v, \varphi \rangle, \text{ for any } \varphi \in D^{1,2}(\mathbb{R}^N). \quad \square$$

**Lemma 3.4.** ([22], A corollary of Theorem 1) *If  $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$  is bounded, then, up to a subsequence, either  $u_n \xrightarrow{n} 0$  in  $L^{2^*}(\mathbb{R}^N)$  or there exist  $\{s_n\} \subset (0, \infty)$  and  $\{\xi_n\} \subset \mathbb{R}^N$  such that*

$$T_{s_n, \xi_n} u_n \xrightarrow{n} u \neq 0 \text{ weakly in } L^{2^*}(\mathbb{R}^N).$$

Let

$$D_s^{1,2}(\mathbb{R}^N) \triangleq \{u \in D^{1,2}(\mathbb{R}^N) : u(x) = u(y, z) = u(|y|, z)\},$$

we see that  $D_s^{1,2}(\mathbb{R}^N) \subset D^{1,2}(\mathbb{R}^N)$  is a closed set, hence  $D_s^{1,2}(\mathbb{R}^N)$  is a Hilbert space with scalar product as (3.2). Based on Lemmas 3.1 to 3.4, we have the following lemma which ensures us to get a nontrivial solution for (1.1) without proving the (PS) condition.

**Lemma 3.5.** *If  $\{u_n\} \subset D_s^{1,2}(\mathbb{R}^N)$  is bounded and there exist  $\{s_n\} \subset (0, +\infty)$  and  $\{x_n\} \subset \mathbb{R}^N$  with  $x_n = (y_n, z_n) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$  such that*

$$T(s_n, x_n)u_n \xrightarrow{n} u \neq 0 \text{ weakly in } L^{2^*}(\mathbb{R}^N). \quad (3.9)$$

Then

$$v_n = T(s_n, 0)w_n \xrightarrow{n} v \neq 0 \text{ weakly in } D_s^{1,2}(\mathbb{R}^N),$$

where  $w_n = T(1, \tilde{z}_n)u_n$  and  $\tilde{z}_n = (0, z_n)$ . Moreover, if  $\{u_n\}$  is also bounded in  $L^q(\mathbb{R}^N)$  for some  $1 < q < 2^*$ , then, there exists constant  $l > 0$  such that  $s_n > l$  for all  $n$ .

**Proof:** The proof of this lemma is almost the same as that of Lemma 23 in [5]. But for the sake of completeness, we give its proof.

Since  $\{u_n\}$  is bounded in  $D_s^{1,2}(\mathbb{R}^N)$ , by the definition of  $T_{s, \xi}$  we see that  $\{v_n\}$  is also bounded in  $D_s^{1,2}(\mathbb{R}^N)$ . Then there is  $v \in D_s^{1,2}(\mathbb{R}^N)$  such that

$$v_n = T(s_n, 0)w_n \xrightarrow{n} v \text{ weakly in } D_s^{1,2}(\mathbb{R}^N).$$

We claim that  $v \neq 0$ . Otherwise if  $v \equiv 0$ , then it leads to a contradiction in the following two cases. For  $x_n = (y_n, z_n)$ , we note that

$$\tilde{y}_n = (y_n, 0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \quad \tilde{z}_n = (0, z_n) \in \mathbb{R}^k \times \mathbb{R}^{N-k}.$$

**Case A:** If  $\{s_n \tilde{y}_n\} \subset \mathbb{R}^N$  is bounded. Then, there is  $\tilde{y}_0 = (y_0, 0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$  such that  $s_n \tilde{y}_n \xrightarrow{n} \tilde{y}_0$  and from (3.3) we have

$$T_{1, -s_n \tilde{y}_n} T_{s_n, \tilde{y}_n} w_n = T_{1, -s_n \tilde{y}_n} T_{s_n, x_n} u_n \xrightarrow{n} T_{1, -\tilde{y}_0} u \neq 0 \quad \text{in } L^{2^*}(\mathbb{R}^N),$$

where we have used assumption (3.9) and Lemma 3.2. On the other hand, since  $v \equiv 0$ , from (3.3) we have

$$T_{1, -s_n \tilde{y}_n} T_{s_n, \tilde{y}_n} w_n = T_{s_n, 0} w_n = v_n \xrightarrow{n} 0 \quad \text{in } D^{1,2}(\mathbb{R}^N),$$

then we have a contradiction.

**Case B:** If  $|s_n \tilde{y}_n| \rightarrow +\infty$ . We claim that there is also a contradiction. Indeed, since  $u \neq 0$ , there exist

$\Omega \subset \mathbb{R}^N$ ,  $|\Omega| \neq 0$  and  $\kappa > 0$  such that  $u > \kappa$  or  $u < -\kappa$  a.e in  $\Omega$ . So we can choose  $R > 0$  such that  $|B_R \cap \Omega| > 0$  and

$$\left| \int_{\mathbb{R}^N} T_{s_n, \tilde{y}_n} w_n \chi_{B_R \cap \Omega} dx \right| \xrightarrow{n} \left| \int_{\mathbb{R}^N} u \chi_{B_R \cap \Omega} dx \right| \geq \kappa |B_R \cap \Omega| > 0.$$

But,

$$T_{s_n, \tilde{y}_n} w_n = T_{s_n, \tilde{y}_n} T_{s_n^{-1}, 0} v_n = T_{1, s_n \tilde{y}_n} v_n.$$

Then,

$$\begin{aligned} \left| \int_{\mathbb{R}^N} T_{s_n, \tilde{y}_n} w_n \chi_{B_R \cap \Omega} dx \right| &\leq \int_{B_R} |T_{s_n, \tilde{y}_n} w_n| dx \\ &= \int_{B_R(s_n \tilde{y}_n)} |v_n| dx \\ &\leq C_R \left\{ \int_{B_R(s_n \tilde{y}_n)} |v_n|^{2^*} dx \right\}^{\frac{1}{2^*}}. \end{aligned}$$

This implies

$$\inf_n \int_{B_R(s_n \tilde{y}_n)} |v_n|^{2^*} dx > \epsilon > 0.$$

Since  $|s_n \tilde{y}_n| \rightarrow +\infty$ , by Lemma 3.2 we have that for any  $m \in \mathbb{N}$  we have  $\{g_i\}_{i=1}^m \subset O(N)$  and  $n_m \in \mathbb{N}$  such that

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n|^{2^*} dx &= \int_{\mathbb{R}^N} |v_n|^{2^*} dx \geq \sum_{i=1}^m \int_{B_R(g_i(s_n \tilde{y}_n))} |v_n|^{2^*} dx \\ &= m \int_{B_R(s_n \tilde{y}_n)} |v_n|^{2^*} dx > m\epsilon \quad \text{for } n > n_m, \end{aligned}$$

where we have used (3.4) and  $v(y, z) = v(|y|, z)$ . Let  $m \rightarrow \infty$ , we have  $\|u_n\|_{2^*} \xrightarrow{n} +\infty$ , which contradicts that  $\{u_n\} \subset L^{2^*}$  is bounded.

Now we can choose  $\varphi \in C_0^\infty(\mathbb{R}^N)$  satisfying  $\int_{\mathbb{R}^N} v \varphi dx \neq 0$ . Choose  $R > 0$  such that  $\text{supp} \varphi \subset B_R$ . Since  $u \in D^{1,2}(\mathbb{R}^N) \rightarrow T(s, \xi)u \in D^{1,2}(\mathbb{R}^N)$  is isometric, we obtain  $\{T_{\lambda_n, 0} w_n\}$  is bounded in  $D^{1,2}(B_R)$ , so is in  $L^2(B_R)$ , hence  $T_{s_n, 0} w_n \rightharpoonup v$  in  $L^2(B_R)$ . Then we have

$$\int_{\mathbb{R}^N} T_{s_n, 0} w_n \varphi dx = \int_{B_R} T_{s_n, 0} w_n \varphi dx \rightarrow \int_{B_R} v \varphi dx = \int_{\mathbb{R}^N} v \varphi dx \neq 0$$

On the otherhand we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} T_{s_n, 0} w_n \varphi dx \right| &\leq \|\varphi\|_\infty |B_R|^{\frac{q-1}{q}} \|T_{s_n, 0} w_n\|_{L^q(B_R)} \\ &\leq s_n^{\frac{N}{q} - \frac{N-2}{2}} \|\varphi\|_\infty |B_R|^{\frac{q-1}{q}} \sup_n \|u_n\|_q. \end{aligned}$$

Since  $1 < q < 2^*$ ,  $\frac{N}{q} - \frac{N-2}{2} > 0$ . So, if  $\lim_{n \rightarrow \infty} s_n = 0$ , we obtain a contradiction. This implies that there exists  $l > 0$  such that  $\inf_n s_n > l$ , since  $s_n > 0$  for all  $n$ .  $\square$

**Lemma 3.6.** *Let  $u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}$  be a nonnegative function, and  $K \subset \mathbb{R}^N$  be a closed set with zero measure, Then there exists  $\varphi \in C_0^\infty(\mathbb{R}^N \setminus K)$  with  $\varphi \geq 0$  such that  $\int_{\mathbb{R}^N} \nabla u \nabla \varphi dx > 0$ .*

*Proof.* Since  $K \subset \mathbb{R}^N$  is closed and  $u \neq 0$ , we can choose a ball  $B \subset \subset \mathbb{R}^N \setminus K$ , and a nonnegative function  $f \in C_0^\infty(B) \subset C_0^\infty(\mathbb{R}^N \setminus K)$  such that  $\int_{\mathbb{R}^N} u f dx > 0$ . Otherwise, we should have that  $u(x) = 0$

a.e in  $x \in \mathbb{R}^N \setminus K$ , and it follows from  $|K| = 0$  that  $u(x) = 0$  a.e in  $x \in \mathbb{R}^N$ , which contradicts  $u \neq 0$  in  $D^{1,2}(\mathbb{R}^N)$ . Then the problem

$$\begin{cases} -\Delta v = f, & x \in B \\ v = 0, & x \in \partial B \end{cases}$$

has a nontrivial solution  $\tilde{\varphi} \geq 0$  on  $B$  and  $\tilde{\varphi} \in C_0^\infty(B)$ . Setting

$$\varphi = \begin{cases} \tilde{\varphi}, & x \in B \\ 0, & x \in \mathbb{R}^N \setminus B. \end{cases}$$

Hence,

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi dx = \int_{\mathbb{R}^N} u f dx > 0$$

□

Based on Lemmas 3.5 and 3.6, we prove now the following theorem, which is important for proving our main Theorems 1.1 and 1.2.

**Theorem 3.1.** *Let  $\{u_n\} \subset E$  be nonnegative sequence such that  $\|u_n\|_E + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \leq C$  and*

$$\int_{\mathbb{R}^3} [\nabla u_n \nabla \varphi + (\frac{1}{|y|^\alpha} + \lambda_n) u_n \varphi] dx + \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n \varphi dx = \int_{\mathbb{R}^3} u_n^p \varphi dx + o(1), \quad (3.10)$$

for any  $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$ , where  $\alpha \geq 0$ ,  $p \in (2, 5)$  and  $\lambda_n \geq 0$  with  $\lambda_n \xrightarrow{n} \lambda_0 < +\infty$ . If  $\{u_n\}$  does not converge to 0 in  $L^6(\mathbb{R}^3)$ , then there exist  $\{\tilde{z}_n\} = \{(0, z_n)\} \subset \mathbb{R}^2 \times \mathbb{R}$  and nonnegative function  $w \in E \setminus \{0\}$  such that

$$w_n = T_{1, \tilde{z}_n} u_n \xrightarrow{n} w \text{ weakly in } E,$$

and

$$\int_{\mathbb{R}^3} [\nabla w \nabla \varphi + (\frac{1}{|y|^\alpha} + \lambda_0) w \varphi] dx + \int_{\mathbb{R}^3} \phi_w(x) w \varphi dx = \int_{\mathbb{R}^3} w^p \varphi dx, \quad (3.11)$$

for any  $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$ . Moreover,  $\|w\|_E + \int_{\mathbb{R}^3} \phi_w w^2 dx \leq C$  and  $w \in C^2(\mathbb{R}^3 \setminus T)$ .

*Proof.* If  $\{u_n\} \subset E$  does not converges to 0 in  $L^6(\mathbb{R}^3)$ , by Lemma 3.4 with  $N = 3$ , there exist  $\{s_n\} \subset (0, +\infty)$  and  $\{x_n\} \subset \mathbb{R}^3$  with  $x_n = (y_n, z_n) \in \mathbb{R}^2 \times \mathbb{R}$  such that

$$T_{s_n, x_n} u_n \xrightarrow{n} u \neq 0 \text{ weakly in } L^6(\mathbb{R}^3). \quad (3.12)$$

Let

$$\tilde{z}_n = (0, z_n) \in \mathbb{R}^2 \times \mathbb{R}^1, \quad w_n = T_{1, \tilde{z}_n} u_n = T(1, \tilde{z}_n) u_n(x). \quad (3.13)$$

By (3.12) and Lemma 3.5 with  $N = 3$ , we have that

$$v_n = T_{s_n, 0} w_n \xrightarrow{n} v \neq 0, \text{ weakly in } D_s^{1,2}(\mathbb{R}^3), \quad (3.14)$$

where  $v$  is nonnegative. And we claim that  $s_n > l > 0$  for all  $n \in \mathbb{N}$ . Indeed, since  $-\Delta \phi_{u_n} = u_n^2$ , we easily conclude

$$\int_{\mathbb{R}^3} |u_n|^3 dx = \int_{\mathbb{R}^3} \nabla \phi_{u_n} \nabla u_n dx \text{ and } \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2 dx.$$

By using Hölder inequality, we deduce that

$$2 \int_{\mathbb{R}^3} |u_n|^3 dx \leq \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2 dx = \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \leq C.$$

So, by using Lemma 3.5 with  $N = 3$  and  $q = 3$ , there is  $l > 0$  such that  $s_n > l$  for all  $n \in \mathbb{N}$ .

**Step1:** There exists  $L > l > 0$  such that  $s_n < L$  for  $n \in \mathbb{N}$  large.

Recalling the definition of  $T$  in (1.11), we have  $|T| = 0$ . Since  $v \geq 0$ , by Lemma 3.6, we have a nonnegative function  $\varphi_1 \in C_0^\infty(\mathbb{R}^3 \setminus T)$  such that

$$\int_{\mathbb{R}^3} \nabla v \nabla \varphi_1 dx > 0.$$

It follows from (3.13) and (3.14) that

$$\int_{\mathbb{R}^3} (\nabla(T_{s_n, \tilde{z}_n} u_n) \nabla \varphi_1) dx \rightarrow \int_{\mathbb{R}^3} \nabla v \nabla \varphi_1 dx > 0. \quad (3.15)$$

Noting that  $T_{s_n, \tilde{z}_n}^{-1} \varphi_1(x) = s_n^{\frac{1}{2}} \varphi_1(s_n x - s_n \tilde{z}_n)$ , then  $T_{s_n, \tilde{z}_n}^{-1} \varphi_1(x) \in C_0^\infty(\mathbb{R}^3 \setminus T)$ , by (3.10), as  $n \rightarrow +\infty$ , we have that

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{u_n} u_n T_{s_n, \tilde{z}_n}^{-1} \varphi_1 dx &+ \int_{\mathbb{R}^3} [\nabla u_n \nabla (T_{s_n, \tilde{z}_n}^{-1} \varphi_1) + (\lambda_n + \frac{1}{|y|^\alpha}) u_n T_{s_n, \tilde{z}_n}^{-1} \varphi_1] dx \\ &= \int_{\mathbb{R}^3} u_n^p T_{s_n, \tilde{z}_n}^{-1} \varphi_1 dx + o(1). \end{aligned}$$

It follows from  $u_n \geq 0$  and  $\lambda_n \geq 0$  that

$$\int_{\mathbb{R}^3} \nabla u_n \nabla (T_{s_n, \tilde{z}_n}^{-1} \varphi_1) dx \leq \int_{\mathbb{R}^3} u_n^p T_{s_n, \tilde{z}_n}^{-1} \varphi_1 dx + o(1).$$

That is

$$\begin{aligned} \int_{\mathbb{R}^3} (\nabla(T_{s_n, \tilde{z}_n} u_n) \nabla \varphi_1) dx &\leq s_n^{\frac{p-5}{2}} \int_{\mathbb{R}^3} (T_{s_n, \tilde{z}_n} u_n)^p \varphi_1 dx + o(1) \\ &\leq C s_n^{\frac{p-5}{2}} \int_{\text{supp} \varphi_1} (T_{s_n, \tilde{z}_n} u_n)^p dx + o(1) \\ &\leq C s_n^{\frac{p-5}{2}} \|T_{s_n, \tilde{z}_n} u_n\|_6^p + o(1) \text{ for } 2 < p < 5 \\ &\leq C s_n^{\frac{p-5}{2}} \|\nabla u_n\|_2^p + o(1) \quad \text{by (3.1)}. \end{aligned}$$

Since  $\{u_n\}$  is bounded in  $E$ , if  $s_n \rightarrow \infty$ , it follows that  $\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} (\nabla(T_{s_n, \tilde{z}_n} u_n) \nabla \varphi_1) dx \leq 0$ , which contradicts with (3.15).

**Step 2:**  $\{u_n\}$  is a bounded sequence in  $E$  such that for any  $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$ , as  $n \rightarrow +\infty$ ,

$$\int_{\mathbb{R}^3} [\nabla w_n \nabla \varphi + (\frac{1}{|y|^\alpha} + \lambda_n) w_n \varphi] dx + \int_{\mathbb{R}^3} \phi_{w_n}(x) w_n \varphi dx = \int_{\mathbb{R}^3} w_n^p \varphi dx + o(1). \quad (3.16)$$

By the definition of  $T_{s, \xi}$  in (3.3), we have

$$\|\nabla(T_{1, \tilde{z}_n} u_n)\|_2 = \|\nabla u_n\|_2, \quad \int_{\mathbb{R}^3} \frac{|T_{1, \tilde{z}_n} u_n|^2}{|y|^\alpha} dx = \int_{\mathbb{R}^3} \frac{|u_n|^2}{|y|^\alpha} dx,$$

hence,  $\|w_n\|_E^2 = \|T_{1, \tilde{z}_n} u_n\|_E^2 = \|u_n\|_E^2$  and  $\{w_n\}$  is bounded in  $E$ . By the definitions of  $T_{1, \tilde{z}_n}$  in (3.3) and  $\phi_u$  in (1.7), it is easy to see that

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{u_n} u_n T_{1, \tilde{z}_n}^{-1} \varphi dx &= \int_{\mathbb{R}^3} T_{1, \tilde{z}_n}(\phi_{u_n} u_n) \varphi dx = \int_{\mathbb{R}^3} \phi_{w_n} w_n \varphi dx, \\ \int_{\mathbb{R}^3} [\nabla u_n \nabla T_{1, \tilde{z}_n}^{-1} \varphi + (\frac{1}{|y|^\alpha} + \lambda_n) u_n T_{1, \tilde{z}_n}^{-1} \varphi] dx &= \int_{\mathbb{R}^3} [\nabla w_n \nabla \varphi + (\frac{1}{|y|^\alpha} + \lambda_n) w_n \varphi] dx, \end{aligned}$$

and

$$\int_{\mathbb{R}^3} u_n^p T_{1, \tilde{z}_n}^{-1} \varphi dx = \int_{\mathbb{R}^3} w_n^p \varphi dx.$$



Those and (3.10) imply that (3.16) holds.

**Step 3:**  $w_n \xrightarrow{n} w \not\equiv 0$  in  $E$  and  $w(x) \geq 0$  a.e. in  $x \in \mathbb{R}^3$ .

By Step 1, there exists  $s_0 \in [l, L]$  such that, passing to subsequence,  $s_n \xrightarrow{n} s_0$ . Then, it follows from (3.14) and Lemma 3.3 that

$$w_n = T_{s_n,0}^{-1} v_n \xrightarrow{n} T_{s_0,0}^{-1} v \not\equiv 0 \quad \text{weakly in } D_s^{1,2}(\mathbb{R}^3). \quad (3.17)$$

By Step 2, there exists  $w \in E$  such that, passing to a subsequence,  $w_n \xrightarrow{n} w$  weakly in  $E$ , since  $E \subset D_s^{1,2}(\mathbb{R}^3)$ , we have  $(D_s^{1,2}(\mathbb{R}^3))^* \subset E^*$ , hence  $w_n \xrightarrow{n} w$  weakly in  $D_s^{1,2}(\mathbb{R}^3)$ , it follows from (3.17) that  $w = T_{s_0,0}^{-1} v \not\equiv 0$  and  $w(x) \geq 0$  a.e. in  $x \in \mathbb{R}^3$ , since  $v \geq 0$  in (3.14).

**Step 4:**  $\phi_w \in D^{1,2}(\mathbb{R}^3)$  and (3.11) holds.

For each  $n \in \mathbb{N}$ ,  $\|\nabla \phi_{w_n}\|_2^2 = \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx = \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx$ , hence,  $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx < C$  implies that  $\{\phi_{w_n}\}$  is bounded in  $D_s^{1,2}(\mathbb{R}^3)$ . So, there exists  $\phi \in D_s^{1,2}(\mathbb{R}^3)$  such that  $\phi_{w_n} \xrightarrow{n} \phi$  weakly in  $D_s^{1,2}(\mathbb{R}^3)$ , that is

$$\int_{\mathbb{R}^3} \nabla \phi_{w_n} \nabla \varphi dx \xrightarrow{n} \int_{\mathbb{R}^3} \nabla \phi \nabla \varphi dx, \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^3). \quad (3.18)$$

On the other hand, for any  $\varphi \in C_0^\infty(\mathbb{R}^3)$ , we have

$$\int_{\mathbb{R}^3} \nabla \phi_{w_n} \nabla \varphi dx = \int_{\mathbb{R}^3} w_n^2 \varphi dx \quad \text{and} \quad \int_{\mathbb{R}^3} w_n^2 \varphi dx \xrightarrow{n} \int_{\mathbb{R}^3} w^2 \varphi dx. \quad (3.19)$$

It follows from (3.18) and (3.19) that

$$\int_{\mathbb{R}^3} \nabla \phi \nabla \varphi dx = \int_{\mathbb{R}^3} w^2 \varphi dx \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^3).$$

So,  $\phi$  is a solution of  $-\Delta \phi = w^2$  in the sense of distribution. Since  $w \in E \subset L^6(\mathbb{R}^3)$ ,  $\phi_w(x) = \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy \in W^{2,3}(\mathbb{R}^3)$  by Theorem 9.9 in [16], hence  $\phi_w$  satisfies  $-\Delta \phi_w = w^2$  in the sense of distribution (Theorem 6.21 in [19]). By uniqueness, we have  $\phi_w = \phi \in D_s^{1,2}(\mathbb{R}^3)$ . It follows from (3.18) that

$$\phi_{w_n} \xrightarrow{n} \phi_w \quad \text{weakly in } D_s^{1,2}(\mathbb{R}^3).$$

Then (see (3.18) in [17] for the details), for any  $\varphi \in C_0^\infty(\mathbb{R}^3)$ , we have

$$\int_{\mathbb{R}^3} \phi_{w_n}(x) w_n \varphi dx \xrightarrow{n} \int_{\mathbb{R}^3} \phi_w(x) w \varphi dx.$$

For each bounded domain  $\Omega \subset \mathbb{R}^3$  and  $q \in (1, 6)$ , it follows from (3.18) and the compactness of Sobolev embedding that  $w_n \xrightarrow{n} w$  strongly in  $L^q(\Omega)$ . Hence, for any  $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$ ,

$$\int_{\mathbb{R}^3} (\nabla w_n \nabla \varphi + (\frac{1}{|y|^\alpha} + \lambda_n) w_n \varphi) dx \xrightarrow{n} \int_{\mathbb{R}^3} (\nabla w \nabla \varphi + (\frac{1}{|y|^\alpha} + \lambda_0) w \varphi) dx$$

and

$$\int_{\mathbb{R}^3} w_n^p \varphi dx \xrightarrow{n} \int_{\mathbb{R}^3} w^p \varphi dx.$$

Those and (3.16) imply that (3.11) holds.

**Step 5.**  $\|w\|_E + \int_{\mathbb{R}^3} \phi_w w^2 dx < C$ .

By Step 3, we have  $w_n \xrightarrow{n} w$  weakly in  $E$ , and Step 4 implies that

$$\int_{\mathbb{R}^3} \phi_w w^2 dx = \|\nabla \phi_w\|_2^2 \quad \text{and} \quad \phi_{w_n} \xrightarrow{n} \phi_w \quad \text{weakly in } D^{1,2}(\mathbb{R}^3),$$

and the lower semi-continuity of norm implies that

$$\|w\|_E \leq \liminf_{n \rightarrow +\infty} \|w_n\|_E,$$

and

$$\int_{\mathbb{R}^3} \phi_w w^2 dx = \|\nabla \phi_w\|_2^2 \leq \liminf_{n \rightarrow +\infty} \|\nabla \phi_{w_n}\|_2^2 = \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx.$$

Hence, by (3.13), we have

$$\begin{aligned} \|w\|_E + \int_{\mathbb{R}^3} \phi_w w^2 dx &\leq \liminf_{n \rightarrow +\infty} \left\{ \|w_n\|_E + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx \right\} \\ &= \liminf_{n \rightarrow +\infty} \left\{ \|u_n\|_E + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \right\} \leq C. \end{aligned}$$

**Step 6.**  $w(x) \in C^2(\mathbb{R}^3 \setminus T)$ .

Since  $\lambda_0 \geq 0$  and  $w(x) \geq 0$  for a.e.  $x \in \mathbb{R}^3$ , it follows from (3.11) that, for any nonnegative function  $v \in C^\infty(\mathbb{R}^3 \setminus T)$ ,

$$\int_{\mathbb{R}^3} \nabla w \nabla v dx \leq \int_{\mathbb{R}^3} w^p v dx. \quad (3.20)$$

Then, Lemma 4.2 in section 4 implies that (3.20) holds also for any nonnegative function  $v \in H^1(\mathbb{R}^3)$ . Note that, for any nonnegative function  $\varphi \in C_0^\infty(\mathbb{R}^3)$  and any nonnegative piecewise smooth function  $h$  on  $[0, +\infty)$ ,  $h(w)\varphi \in H^1(\mathbb{R}^3)$ . Take  $v = h(w)\varphi$  in (3.20), then we see that (4.7) in section 4 holds with  $u = w$  and  $N = 3$ . Hence, by Lemma 4.3, we have  $w \in L^\infty(\mathbb{R}^3)$ . Let  $\Omega \subset\subset \mathbb{R}^3 \setminus T$  be a bounded domain with smooth boundary, then  $\frac{1}{|y|}$  is a smooth function in  $\Omega$  and  $w \in W^{1,2}(\Omega)$  is a weak solution of

$$-\Delta w(x) = f(x), \quad x \in \Omega, \quad (3.21)$$

where  $f(x) = |w|^{p-1}w(x) - \phi_w(x)w(x) - (\lambda_0 + \frac{1}{|y|})w(x)$ . Since  $w, \phi_w \in W^{1,2}(\Omega)$  and  $w \in L^\infty(\Omega)$ , we have  $f(x) \in W^{1,2}(\Omega)$ . By using Theorem 8.10 in [16], we get  $w \in W_{loc}^{3,2}(\Omega)$ . Then, Sobolev imbedding theorem implies that  $w \in C_{loc}^{1/4}(\Omega)$ , hence  $\phi_w(x) \in C_{loc}^{2,1/4}(\Omega)$  since  $\phi_w(x)$  is a weak solution of  $-\Delta \phi(x) = w^2(x)$  in  $D^{1,2}(\Omega)$ . It follows that  $f(x) \in C_{loc}^{1/4}(\Omega)$ . By applying Theorem 9.19 in [16] to (3.21), we have  $w \in C_{loc}^{2,1/4}(\Omega)$ . So  $w \in C^2(\mathbb{R}^3 \setminus T)$ .  $\square$

**Proof of Theorem 1.1** Let  $\{u_n\} \subset H$  be the bounded nonnegative (PS) sequence obtained by Lemma 2.6, then there exists  $C > 0$ , which is independent of  $\lambda$  if  $\lambda \in (0, 1)$ , such that

$$\|u_n\|_H^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \leq C \text{ and } u_n(x) \geq 0 \text{ a.e. in } x \in \mathbb{R}^3. \quad (3.22)$$

Hence,

$$\|u_n\|_E^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \leq C, \quad u_n(x) \geq 0 \text{ a.e. in } x \in \mathbb{R}^3.$$

And (2.20) implies that (3.10) holds with  $\lambda_n \equiv \lambda > 0$ . If  $\{u_n\}$  does not converges to 0 in  $L^6(\mathbb{R}^3)$ , by Theorem 3.1, there exist  $\{\tilde{z}_n\} = \{(0, z_n)\} \subset \mathbb{R}^2 \times \mathbb{R}$  and nonnegative function  $w \in E \setminus \{0\}$  such that

$$w_n = T_{1, \tilde{z}_n} u_n \xrightarrow{n} w \text{ weakly in } E, \quad (3.23)$$

$$\int_{\mathbb{R}^3} [\nabla w \nabla \varphi + (\frac{1}{|y|^\alpha} + \lambda)w\varphi] dx + \int_{\mathbb{R}^3} \phi_w(x)w\varphi dx = \int_{\mathbb{R}^3} w^p \varphi dx,$$

for any  $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$ , i.e.,  $w$  is a weak solution of (1.1) in  $E$ . Moreover,  $w \in C^2(\mathbb{R}^3 \setminus T)$  and

$$\|w\|_E + \int_{\mathbb{R}^3} \phi_w w^2 dx \leq C. \quad (3.24)$$

Now, we claim that  $w \in H$ . In fact, by (3.3) and (3.23), we have  $\|w_n\|_H = \|u_n\|_H$  and  $\|w_n\|_H$  is bounded, so there exists  $w^* \in H$  such that

$$w_n \xrightarrow{n} w^* \text{ weakly in } H \text{ and } w_n(x) \xrightarrow{n} w^*(x), \text{ a.e. in } x \in \mathbb{R}^3. \quad (3.25)$$

On the other hand, (3.23) implies that

$$w_n(x) \xrightarrow{n} w(x), \text{ a.e. in } x \in \mathbb{R}^3.$$

This and (3.25) show that  $w = w^* \in H$ . Moreover, if  $\lambda \in (0, 1)$ , Lemma 2.6 shows that there exists  $M > 0$  independent of  $\lambda \in (0, 1)$  such that (3.22) holds with  $C = M$ , then (3.24) holds with  $C = M$ . Hence, to complete the proof of Theorem 1.1, we only need to prove that  $\{u_n\}$  cannot converges to 0 in  $L^6(\mathbb{R}^3)$ . For  $r \in (2, 6)$ , by Hölder inequality we have

$$\int_{\mathbb{R}^3} |u_n|^r dx = \int_{\mathbb{R}^3} |u_n|^{\frac{2}{q}} |u_n|^{\frac{6}{q'}} dx \leq \|u_n\|_2^{\frac{2}{q}} \|u_n\|_6^{\frac{6}{q'}}$$

where  $q = \frac{4}{6-r} > 1$ ,  $q' = \frac{q}{q-1} = \frac{4}{r-2} > 1$ . Hence, if  $u_n \xrightarrow{n} 0$  in  $L^6(\mathbb{R}^3)$ , then  $u_n \xrightarrow{n} 0$  in  $L^r(\mathbb{R}^3)$  for  $r \in (2, 6)$ , this and (1.8) imply that  $\int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 dx \xrightarrow{n} 0$ . Therefore, by (2.20) we have that, for  $p \in (2, 5)$ ,

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} \left[ I(u_n) - \frac{1}{2} I'(u_n) u_n \right] \\ &= \lim_{n \rightarrow \infty} \left[ -\frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 dx + \frac{p-3}{2(p+1)} \int_{\mathbb{R}^3} |u_n|^{p+1} dx \right] = 0, \end{aligned}$$

this is impossible since  $c > 0$ .  $\square$

## 4 Existence for $\lambda = 0$ : Proof of Theorem 1.2.

We need more lemmas as follows to prove Theorem 1.2. For any  $N \geq 3$  and domain  $\Omega \subset \mathbb{R}^N$  ( $\Omega$  can be bounded or unbounded), let  $\Gamma \subset \Omega$  be a closed Manifold with  $\text{codim} \Gamma = k \geq 2$ . Then,

**Lemma 4.1.**  $C_0^\infty(\Omega \setminus \Gamma)$  is dense in  $H_0^1(\Omega)$ .

**Proof:** For each  $u \in H_0^1(\Omega) \cap C_0^\infty(\Omega \setminus \Gamma)^\perp$  and  $\tilde{\varphi} \in C_0^\infty(\Omega \setminus \Gamma)$ , we have

$$\langle u, \tilde{\varphi} \rangle_{H_0^1(\Omega)} = 0, \quad (4.1)$$

since  $C_0^\infty(\Omega \setminus \Gamma)$  is dense in  $H_0^1(\Omega \setminus \Gamma)$ , it follows that

$$\langle u, \psi \rangle_{H_0^1(\Omega)} = 0 \text{ for any } \psi \in H_0^1(\Omega \setminus \Gamma). \quad (4.2)$$

It is true that  $C_0^\infty(\Omega \setminus \Gamma)$  is dense in  $H_0^1(\Omega)$  if  $C_0^\infty(\Omega \setminus \Gamma)^\perp \cap H_0^1(\Omega) = \{0\}$ . Hence, we only need to show that (4.1) holds with all  $\tilde{\varphi} \in C_0^\infty(\Omega)$  as follows.

For any  $\varphi \in C_0^\infty(\Omega)$ , let  $\Omega_0 = \text{supp} \varphi$ . If  $\Omega_0 \cap \Gamma = \emptyset$ , then  $\varphi \in C_0^\infty(\Omega \setminus \Gamma)$  and (4.1) holds with  $\tilde{\varphi} = \varphi$ . Otherwise  $\Omega_0 \cap \Gamma \neq \emptyset$ , setting  $\Gamma_0 = \Omega_0 \cap \Gamma$ , and for any  $d > 0$  small enough that we have set  $\Gamma_d := \{x \in \Omega : \text{dist}(x, \Gamma_0) < d\} \subset \Omega$ . Let

$$\psi_d(x) := \begin{cases} \frac{\text{dist}(x, \Gamma_{2d})}{d}, & x \in \Gamma_{3d}, \\ 1, & x \in \Omega \setminus \Gamma_{3d}, \end{cases}$$

then  $\psi_d(x) \in C^{0,1}(\Omega)$  and  $\|\psi_d\|_{C^{0,1}(\Omega)} \leq \frac{1}{d}$ . Let  $\varphi_d := \varphi(1 - \psi_d)$ , we have  $\varphi\psi_d \in H_0^1(\Omega \setminus \Gamma)$  and  $\varphi_d \in H_0^1(\Gamma_{3d})$ . It follows from (4.2) that

$$\begin{aligned} \langle u, \varphi \rangle_{H^1} &= \langle u, \varphi_d + \varphi\psi_d \rangle_{H^1} = \langle u, \varphi_d \rangle_{H^1} + \langle u, \varphi\psi_d \rangle_{H^1} \\ &= \langle u, \varphi_d \rangle_{H^1} \leq \|u\|_{H^1(\Gamma_{3d})} \|\varphi_d\|_{H^1(\Gamma_{3d})}. \end{aligned} \quad (4.3)$$

By the definition of  $\varphi_d$ , we have

$$\|\varphi_d\|_{L^2(\Gamma_{3d})}^2 = \int_{\Gamma_{3d}} \varphi_d^2 dx \leq 4 \|\varphi\|_{L^\infty(\Omega)}^2 |\Gamma_{3d}| \xrightarrow{d \rightarrow 0} 0, \quad (4.4)$$

$$\begin{aligned} \|\nabla \varphi_d\|_{L^2(\Gamma_{3d})}^2 &= \int_{\Gamma_{3d}} |\nabla \varphi_d|^2 dx \leq C \|\varphi\|_{C^1(\Omega)}^2 |\Gamma_{3d}| (1 + \frac{1}{d^2}) \\ \text{since } k \geq 2 &\leq_{\text{codim } \Gamma = k} C d^{k-2} \leq C. \end{aligned} \quad (4.5)$$

And  $|\Gamma_{3d}| \xrightarrow{d \rightarrow 0} 0$  implies that

$$\|u\|_{H^1(\Gamma_{3d})} \xrightarrow{d \rightarrow 0} 0. \quad (4.6)$$

It follows from (4.3) to (4.6) that (4.1) holds for all  $\tilde{\varphi} \in C_0^\infty(\Omega)$ .  $\square$

**Lemma 4.2.**  $\{\varphi \in H_0^1(\Omega \setminus \Gamma) : \varphi(x) \geq 0\}$  is dense in  $\{\varphi \in H_0^1(\Omega) : \varphi(x) \geq 0\}$ .

**Proof:** Lemma 4.1 shows that for any  $u(x) \in H_0^1(\Omega)$ , there exist  $\{\varphi_n(x)\} \subset C_0^\infty(\Omega \setminus \Gamma)$  such that

$$\|\varphi_n - u\|_{H^1(\Omega)} \xrightarrow{n} 0.$$

This lemma is proved if we have

$$\| |\varphi_n| - |u| \|_{H^1(\Omega)} \xrightarrow{n} 0,$$

which is true by the following two facts,

$$\begin{aligned} 0 &\leq \| |\varphi_n| - |u| \|_2^2 = \int_{\Omega} |\varphi_n|^2 + |u|^2 - 2|\varphi_n||u| dx \\ &\leq \int_{\Omega} \varphi_n^2 + u^2 - 2\varphi_n u dx = \|\varphi_n - u\|_2^2 \xrightarrow{n} 0. \end{aligned}$$

$$\begin{aligned} 0 &\leq \| |\varphi_n| - |u| \|_{D^{1,2}}^2 = \int_{\Omega} |\nabla |\varphi_n||^2 + |\nabla |u||^2 - 2\nabla |\varphi_n| \nabla |u| dx \\ &= \int_{\Omega} |\nabla(\varphi_n - u)|^2 dx + 4 \int_{\Omega} \nabla \varphi_n^+ \nabla u^- + \nabla \varphi_n^- \nabla u^+ dx \xrightarrow{n} 0. \quad \square \end{aligned}$$

**Lemma 4.3.** (Lemma 3.2 of [18]) Let  $N \geq 3$ ,  $p \in (1, \frac{N+2}{N-2})$  and let  $u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}$  be a nonnegative function such that

$$\int_{\mathbb{R}^N} \nabla u \nabla (h(u)\varphi) dx \leq \int_{\mathbb{R}^N} |u|^{p-1} u h(u) \varphi dx, \quad (4.7)$$

holds for any nonnegative  $\varphi \in C_0^\infty(\mathbb{R}^N)$  and any nonnegative piecewise smooth function  $h$  on  $[0, +\infty)$  with  $h' \in L^\infty(\mathbb{R})$ . Then,  $u \in L^\infty(\mathbb{R}^N)$  and there exist  $C_1 > 0$  and  $C_2 > 0$ , which depend only on  $N$  and  $p$ , such that

$$\|u\|_\infty \leq C_1 \left(1 + \|u\|_{2^*}^{C_2}\right) \|u\|_{2^*}.$$

**Lemma 4.4.** For  $p > 2$ , let  $(u, \phi) \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$  be a nontrivial nonnegative weak solution of the following problem

$$\begin{cases} -\Delta u + \mu \phi(x)u \leq |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \quad (4.8)$$

where  $\mu > 0$ . Then

$$\|u\|_\infty > \mu^{\frac{1}{2(p-2)}}.$$

*Proof:* By assumption,  $(u, \phi) \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$  is a weak solution of (4.8), then, for any nonnegative function  $v \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$ , we have

$$\int_{\mathbb{R}^3} \nabla u \nabla v dx + \mu \int_{\mathbb{R}^3} \phi(x) u v dx - \int_{\mathbb{R}^3} |u|^{p-1} u v dx \leq 0, \quad (4.9)$$

$$\int_{\mathbb{R}^3} \nabla \phi \nabla v dx = \int_{\mathbb{R}^3} u^2 v dx. \quad (4.10)$$

For  $c > 0$ , adding  $c \int_{\mathbb{R}^3} u^2 v dx$  to both sides of (4.9), and using (4.10) we get that

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla u \nabla v dx + \int_{\mathbb{R}^3} [cu^2 - |u|^{p-1}u] v dx + \mu \int_{\mathbb{R}^3} \phi(x) u v dx \\ \leq c \int_{\mathbb{R}^3} \nabla \phi \nabla v dx, \quad \text{for any } v \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3). \end{aligned} \quad (4.11)$$

In the following, we mean that  $w_+(x) = \max\{0, w(x)\}$  for any function  $w(x)$  on  $\mathbb{R}^3$ . For the above  $c > 0$ , we let  $\epsilon > 0$  small,

$$w_1(x) = (u(x) - c\phi(x) - \epsilon)^+ \text{ and } \Omega_1 = \{x \in \Omega : w_1(x) > 0\}. \quad (4.12)$$

It is easy to see that  $u(x) \xrightarrow{|x| \rightarrow +\infty} 0$  and  $\phi(x) \geq 0$  a.e.  $x \in \mathbb{R}^3$ , then  $w_1 \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$  and  $u(x)|_{\Omega_1} > c\phi(x) > 0$ . Taking  $v(x) = w_1(x)$  in (4.11), we see that

$$\int_{\Omega_1} \nabla u \nabla w_1 dx + \int_{\Omega_1} [cu^2 - |u|^{p-1}u] w_1 dx \leq c \int_{\Omega_1} \nabla \phi \nabla w_1 dx. \quad (4.13)$$

However, for all  $x \in \Omega_1$  we have  $cu^2 - |u|^{p-1}u \geq 0$  if  $c = \delta^{p-2}$  with  $\delta = \|u\|_\infty$ . Then, let  $c = \delta^{p-2}$  and (4.13) implies that

$$\int_{\Omega_1} \nabla u \nabla w_1 dx - c \int_{\Omega_1} \nabla \phi \nabla w_1 dx \leq 0,$$

that is,

$$\int_{\Omega_1} \nabla (u - \delta^{p-2}\phi) \nabla w_1 dx = \int_{\Omega_1} |\nabla w_1|^2 dx = 0. \quad (4.14)$$

Hence, either  $|\Omega_1| = 0$  or  $w_1|_{\Omega_1} \equiv \text{constant}$ , this means that  $u(x) \leq \delta^{p-2}\phi(x) + \epsilon$  a.e.  $x \in \mathbb{R}^3$ . Let  $\epsilon \rightarrow 0$  we have

$$u(x) \leq \delta^{p-2}\phi(x), \text{ a.e. in } x \in \mathbb{R}^3, \quad (4.15)$$

To prove that  $\|u\|_\infty > \mu^{\frac{1}{2(p-2)}}$ , we let  $v = u$  in (4.9), it follows that

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx + \mu \int_{\mathbb{R}^3} \phi(x) u^2 dx - \int_{\mathbb{R}^3} u^{p+1} dx \leq 0,$$

that is,

$$\mu \int_{\mathbb{R}^3} \phi(x) |u|^2 dx \leq \int_{\mathbb{R}^3} |u|^{p+1} dx.$$

This and (4.15) show that

$$\int_{\mathbb{R}^3} (u^{p-2} - \mu \delta^{2-p}) u^3 dx \geq 0.$$

Hence,  $\delta^{p-2} \geq \mu \delta^{2-p}$  by  $p > 2$ . On the other hand, by using  $u \neq 0$  we have  $\delta > 0$ . Then  $\|u\|_\infty = \delta \geq \mu^{\frac{1}{2(p-2)}}$ .  $\square$

**Proof of Theorem 1.2.** By Theorem 1.1, we know that, for each  $\lambda \in (0, 1)$ , problem (1.1) has nonnegative solution  $u_\lambda \in H \setminus \{0\}$  such that  $\|u_\lambda\|_E + \int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 dx \leq M$  and (3.10) holds with  $u_n = u_\lambda$  and  $\lambda_n = \lambda$ . Since  $u_\lambda \geq 0$ , it follows from (3.10) that

$$\int_{\mathbb{R}^3} \nabla u_\lambda \nabla \varphi dx + \int_{\mathbb{R}^3} \phi_{u_\lambda}(x) u_\lambda \varphi dx \leq \int_{\mathbb{R}^3} u_\lambda^p \varphi dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3 \setminus T), \varphi \geq 0.$$

This and Lemma 4.2 show that

$$\int_{\mathbb{R}^3} \nabla u_\lambda \nabla v dx + \int_{\mathbb{R}^3} \phi_{u_\lambda}(x) u_\lambda v dx \leq \int_{\mathbb{R}^3} u_\lambda^p v dx \quad \text{for all } v \in H^1(\mathbb{R}^3), v \geq 0, \quad (4.16)$$

it follows that (4.8) holds with  $u = u_\lambda$  and  $\mu = 1$ . Hence, by Lemma 4.4, we have

$$\|u_\lambda\|_\infty \geq 1 \text{ for all } \lambda > 0. \quad (4.17)$$

Meanwhile, for any nonnegative function  $\varphi \in C_0^\infty(\mathbb{R}^3)$  and any nonnegative piecewise smooth function  $h$  on  $[0, +\infty)$ , we see that  $h(u_\lambda)\varphi \in H^1(\mathbb{R}^3)$ . Let  $v = h(u_\lambda)\varphi$  in (4.16), it follows that (4.7) holds with  $u = u_\lambda$  and  $N = 3$ . Hence, by Lemma 4.3, we have

$$\|u_\lambda\|_\infty \leq C_1(1 + \|u_\lambda\|_6^{C_2})\|u_\lambda\|_6. \quad (4.18)$$

So, (4.17) and (4.18) imply that  $u_\lambda$  does not converge to 0 in  $L^6(\mathbb{R}^3)$  as  $\lambda \rightarrow 0$ , then Theorems 3.1 shows that there exist nonnegative function  $u \in E$  and  $u \neq 0$  such that,

$$\int_{\mathbb{R}^3} \nabla u \nabla \varphi + \frac{u\varphi}{|y|^\alpha} dx + \int_{\mathbb{R}^3} \phi_u(x) u \varphi dx = \int_{\mathbb{R}^3} u^p \varphi dx, \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^3 \setminus T).$$

Moreover,  $w \in C^2(\mathbb{R}^3 \setminus T)$ .  $\square$

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